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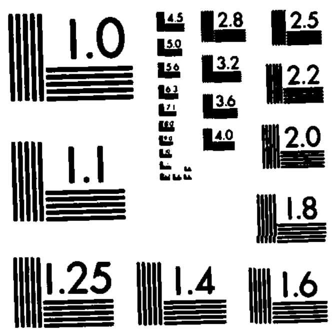

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ROME AIR DEVELOPMENT CENTER

STATISTICAL ANALYSIS OF THE EFFECT OF PHASE QUANTIZATION ON ARRAY ANTENNA SIDELOBES

Robert A. Shore

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ROME AIR DEVELOPMENT CENTER
Air Force Systems Command
Griffiss Air Force Base, NY 13441

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General expressions for the mean and variance of the power at an arbitrary location in the pattern are derived for a linear array of isotropic, equispaced elements. The phases of the element excitation coefficients are assumed to be set by digital phase shifters, causing random phase errors uniformly distributed in the interval of plus to minus the least significant bit of the phase command. Approximations valid for small phase errors ($N_{bit} > 4$) are obtained. For large arrays, the variance of the power at a location where the error-free pattern has a null is approximately equal to the square of the

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mean of the power. An expression is obtained for the probability distribution of the amplitude and power of the error field assuming the error-free field is real and there are sufficient elements in the array for the Central Limit Theorem to be applied. For small phase errors, and points not halfway between grating lobes, the expression reduces to the Rice-Nakagami distribution. It further reduces to the Rayleigh distribution at points where the error-free pattern has a null. Expressions for the pattern power mean and variance for an array of arbitrary size obtained directly are consistent with those obtained by assuming a large number of array elements and using the Central Limit Theorem to derive the probability distribution of the pattern power. The theoretical results are compared with Monte Carlo computer simulations. The Kolmogorov-Smirnov test usually shows close agreement between the simulation and the theoretical results. However, the test also shows that the depths of multiple nulls are not statistically independent when there are multiple nulls in the error-free pattern; the probability distribution of the depth of the least deep null is closer to the distribution predicted for a single null than it is to the distribution for multiple nulls.

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Contents

Illustrations

Illustrations

Tables

Statistical Analysis of the Effect of Phase Quantization on Array Antenna Sidelobes

1. INTRODUCTION

There is currently considerable interest in using phased arrays with digital phase shifters for adaptive nulling. However, the quantized output of a digital phase shifter differs from the ideal value. This difference between the actual and ideal phases is reflected in a corresponding discrepancy between the actual pattern and the ideal or error-free pattern. Of special concern for adaptive nulling applications is the fact that phase quantization errors can cause shallower nulls, reducing the ability of the array to reject interference from particular directions. Therefore, understanding the influence of phase quantization errors on array antenna patterns is important.

A standard technique used to eliminate the spurious quantization lobes, similar to grating lobes, that result from a periodic phase quantization error distribution across the elements of an array, is to randomize the phase shifters by inserting a constant phase shift which differs from element to element.[1] The phase quantization errors can then be regarded as independent of each other. Any general study of the effect of such phase quantization errors on array patterns is necessarily statistical in nature, focusing on statistical parameters of the phase-error induced pattern perturbations such as the mean and variance of the power at a specified

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(Received for publication 7 July 1982)

1. Skolnik, M. I. (1980) Introduction to Radar Systems, McGraw-Hill, New York, p. 321.

pattern location. Since the magnitude of the phase quantization errors has a maximum value of $\pi/2^{N_{bit}}$, where N_{bit} is the number of bits in the phase shifters, it is natural to base a statistical description of the pattern perturbations on the assumption that any particular set of element weight phase errors is obtained by drawing each phase error independently (with replacement) from an underlying population of phase errors uniformly distributed in the interval $[-\pi/2^{N_{bit}}, \pi/2^{N_{bit}}]$.

This report describes the statistical properties of the sidelobes of linear arrays whose element phases are independently and uniformly distributed. The array elements are assumed to be isotropic and equispaced and to have a deterministic but arbitrary amplitude taper. A number of investigations have been devoted to various related aspects of this subject,[2] and in considerable part this report is designed to bring together, for convenient reference and in self-contained form, results that are well established in the literature.

We begin by investigating the mean and variance of the power at an arbitrary pattern location. No conditions are imposed regarding the number of array elements. The resulting expression obtained for the variance of the power is believed to be new. Approximations valid for small phase errors ($N_{bit} > 4$) are obtained from the general expressions for the mean and variance. We then derive an expression for the probability distribution of the amplitude and power of the perturbed field under the assumptions that the error-free field is real and that the number of elements in the array is sufficiently large for the Central Limit Theorem (CLT) to be applied. We specialize this distribution, first to the case of small phase errors and ordinary sidelobe locations (locations not halfway between grating lobes), and then to the important case of sidelobe locations for which the error-free pattern has a null. We show that under the conditions for which the CLT is valid, the exact expression for the variance of the power obtained with no restriction on the number of elements reduces to the expression obtained by means of the CLT and the probability distribution of the power.

The statistical distribution of power at the least deep null among several locations for which the error-free pattern has a null is then obtained by assuming independence of the null depths at two or more locations and multiplying together the probability distributions for the individual nulls. The theoretical analysis is

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2. See for example, Ruze, J. (1952) The effect of aperture errors on the antenna radiation pattern, Nuovo Cimento Suppl., 9(No. 3):364-380; Allen, J. L. (1961) Some extensions of the theory of random error effects on array patterns, Chap. III, Phased Array Radar Studies, 1 July 1960 to 1 July 1961, M.I.T. Lincoln Lab Tech. Rep. 236; Skolnik, M. I. (1980) Nonuniform arrays, Chap. 6 of Antenna Theory, Part 1, Collin, R. E., and Zucker, F. J. (Eds.) McGraw-Hill, N.Y., pp. 318-321; Steinberg, B. D. (1976) Principles of Aperture and Array System Design, Chaps. 8, 9, and 13, John Wiley, N.Y., Shifrin, Y.S. (1971) Statistical Antenna Theory, Chap. 7, Golem Press, Boulder, Colorado.

concluded by deriving the distribution of power at locations halfway between grating lobes. Again we demonstrate consistency between the exact expression for the variance of the power and the expression derived from the CLT under the assumption of large arrays. Following the theoretical analysis, we describe the results of computer simulations performed to compare with theoretical results.

2. ANALYSIS

In this section we derive expressions for the mean and the variance of the power, and the probability distributions of the amplitude and power, at a given direction of the pattern of a linear array of equispaced isotropic radiators, given that the phases of the array elements are subject to random errors. To investigate the effect of phase shifter quantization on the array pattern, we take the mean, variance, and probability distribution over an ensemble of arrays; in each member of the ensemble, we assume the phases of the elements to have mutually independent random deviations δ from their ideal values, uniformly distributed in the interval $\left[-\pi/2^{N_{bit}}, \pi/2^{N_{bit}}\right]$ where N_{bit} is the number of bits in the phase shifters.

Let d be the spacing between the array elements and assume the phase reference center to be the center of the array. Let w_n be the ideal complex weight (that is, the weight in the absence of phase shifter errors) of the nth element, and let δ_n be the corresponding random phase error. Then the error-free field pattern is

$$F_o(u) = \sum_{n=1}^{N} w_n e^{j d_n u}$$

and the pattern in the presence of phase errors is

$$F(u) = \sum_{n=1}^{N} w_n e^{j\delta_n} e^{j d_n u} \tag{1}$$

where

$$d_n = \frac{N-1}{2} - (n-1), \quad n = 1, 2, \ldots, N$$

and

$$u = kd \sin\theta$$

with

$$k = 2\pi/\lambda$$

and θ the angle measured from broadside to the array.

2.1 Mean of the Power

We first obtain an expression for the ensemble average of the power at a given direction of the pattern. The pattern power of a member array is given by

$$|F(u)|^2 = F(u)F^*(u) = \sum_n \sum_m w_n w_m^* e^{j(\delta_n - \delta_m)} e^{j(d_n - d_m)u} \tag{2}$$

so that we seek the ensemble average, $\overline{|F(u)|^2}$. In calculating $\overline{|F(u)|^2}$ we make use of the theorem that the expectation of a product of mutually independent random variables is equal to the product of their expectations.[3] Hence, in the double summation in Eq. (2) we separate the terms for which $n \neq m$ and hence δ_n and δ_m are independent) from those for which $n = m$ (and hence $\delta_n = \delta_m$). For $n = m$, the contribution to the ensemble average is simply

$$\sum_n |w_n|^2$$

while for $n \neq m$ we have

$$\sum_{\substack{n \\ n\neq m}} \sum_m w_n w_m^* \overline{e^{j(\delta_n - \delta_m)}} e^{j(d_n - d_m)u}$$

$$= \left|\overline{e^{j\delta}}\right|^2 \sum_{\substack{n \\ n\neq m}} \sum_m w_n w_m^* e^{j(d_n - d_m)u}$$

where we have made use of the fact that

$$\overline{e^{j(\delta_n - \delta_m)}} = \overline{e^{j\delta_n}}\ \overline{e^{-j\delta_m}} = \left|\overline{e^{j\delta}}\right|^2$$

with δ a random variable uniformly distributed in $\left[-\pi/2^{N_{bit}}, \pi/2^{N_{bit}}\right]$ since the phase errors are assumed to be identically randomly distributed for all elements of the array. The ensemble average of the power is then

3. Feller, W. (1968) An Introduction to Probability Theory and Its Applications, Vol. 1, 3rd Ed., John Wiley, N.Y., pp. 222-230.

$$\overline{|F(u)|^2} = \sum_n |w_n|^2 + \left|\overline{e^{j\delta}}\right|^2 \sum_{\substack{n \\ n \neq m}} \sum_m w_n w_m^* e^{j(d_n - d_m)u}$$

$$= \left|\overline{e^{j\delta}}\right|^2 \sum_n \sum_m w_n w_m^* e^{j(d_n - d_m)u}$$

$$+ \left(1 - \left|\overline{e^{j\delta}}\right|^2\right) \sum_n |w_n|^2 .$$

But

$$\sum_n \sum_m w_n w_m^* e^{j(d_n - d_m)u} = F_o(u) F_o^*(u) = |F_o(u)|^2$$

so that

$$\overline{|F(u)|^2} = \left|\overline{e^{j\delta}}\right|^2 |F_o(u)|^2 + \left(1 - \left|\overline{e^{j\delta}}\right|^2\right) \sum_n |w_n|^2 .$$

The calculation of $\overline{e^{j\delta}}$ is straightforward. For phase errors uniformly distributed in the interval $[-\Delta, \Delta]$, $\Delta = \pi/2^{N_{bit}}$

$$\overline{e^{j\delta}} = \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} e^{j\delta} \, d\delta$$

$$= \frac{\sin\Delta}{\Delta} . \tag{3}$$

Hence

$$\left|\overline{e^{j\delta}}\right|^2 = \left(\frac{\sin\Delta}{\Delta}\right)^2$$

and

$$\overline{|F(u)|^2} = \left(\frac{\sin\Delta}{\Delta}\right)^2 |F_o(u)|^2 + \left[1 - \left(\frac{\sin\Delta}{\Delta}\right)^2\right] \sum_n |w_n|^2 . \tag{4}$$

The ensemble average power pattern is thus the superposition of two terms, the first of which is the error-free power pattern multiplied by a factor whose magnitude is less than unity for $\Delta > 0$, and the second is independent of the direction.

For $\Delta = 0$, $\frac{\sin\Delta}{\Delta} = 1$ and $\overline{|F(u)|^2}$ reduces to $|F_o(u)|^2$. In applications to phase shifters with many bits ($N_{bit} > 4$),

$$\left(\frac{\sin\Delta}{\Delta}\right)^2 \approx 1 - \frac{\Delta^2}{3}$$

and

$$\overline{\left|F(u)\right|^2} \approx \left(1 - \frac{\Delta^2}{3}\right)\left|F_o(u)\right|^2 + \frac{\Delta^2}{3}\sum_n |w_n|^2 \tag{5}$$

At a null of the error-free pattern

$$\overline{|F(u)|^2} = \left[1 - \left(\frac{\sin\Delta}{\Delta}\right)^2\right]\sum_n |w_n|^2, \quad F_o(u) = 0 \tag{6a}$$

$$\approx \frac{\Delta^2}{3}\sum_n |w_n|^2, \quad N_{bit} > 4 . \tag{6b}$$

Note that for each extra bit of the phase shifters, Δ^2 is multiplied by a factor of 1/4 and hence the average power at a null of the error-free pattern decreases by 6 dB. The significance of the filling-in of the nulls of the error-free pattern due to phase errors is usefully assessed in relation to the peak value of the error-free pattern. For error free weights such that the phase varies linearly across the array, the peak power is $\left(\sum_n |w_n|\right)^2$. Dividing Eq. (6a) by this quantity to normalize it, we obtain

$$\frac{\overline{|F(u)|^2}}{\left(\sum_n |w_n|\right)^2} = \left[1 - \left(\frac{\sin\Delta}{\Delta}\right)^2\right]\frac{\sum_n |w_n|^2}{\left(\sum_n |w_n|\right)^2}, \quad F_o(u) = 0$$

$$\approx \frac{\Delta^2}{3}\,\frac{\sum_n |w_n|^2}{\left(\sum_n |w_n|\right)^2}, \quad N_{bit} > 4 .$$

2.2 Variance of the Power

Next we obtain an expression for the variance of the power

$$\mathrm{Var}\left(|F(u)|^2\right) = \overline{\left[|F(u)|^2 - \overline{|F(u)|^2}\right]^2}$$

$$= \overline{\left[|F(u)|^2\right]^2} - \overline{|F(u)|^2}^{\,2} . \tag{7}$$

Since we have already found $\overline{|F(u)|^2}$, we must determine

$$\overline{\left[|F(u)|^2\right]^2} = \sum_n \sum_m \sum_p \sum_q w_n w_m^* w_p w_q^* e^{j(\delta_n - \delta_m + \delta_p - \delta_q)} e^{j(d_n - d_m + d_p - d_q)u} . \tag{8}$$

As above in obtaining an expression for the average power, we proceed by grouping the terms of the quadruple summation according to whether or not equalities hold among the indices. The following cases must be distinguished:

(1) All four indices are equal
$n=m=p=q$

(2) three indices equal
(2a) $n=m=p\neq q$
(2b) $n=m=q\neq p$
(2c) $n=p=q\neq m$
(2d) $m=p=q\neq n$

(3) two pairs of indices equal
(3a) $n=m,\ p=q,\ n\neq p$
(3b) $n=p,\ m=q,\ n\neq m$
(3c) $n=q,\ m=p,\ n\neq m$

(4) two indices equal
(4a) $n=m,\ n\neq p,\ n\neq q,\ p\neq q$
(4b) $n=p,\ n\neq m,\ n\neq q,\ m\neq q$
(4c) $n=q,\ n\neq m,\ n\neq p,\ m\neq p$
(4d) $m=p,\ m\neq n,\ m\neq q,\ n\neq q$
(4e) $m=q,\ m\neq n,\ m\neq p,\ n\neq p$
(4f) $p=q,\ p\neq n,\ p\neq m,\ n\neq m$

(5) no indices equal

We treat each of these cases in turn.

(1) $n=m=p=q$

The contribution of the corresponding terms in the quadruple sum of Eq. (8) is simply

$$\sum_n |w_n|^4$$

(2a) $n=m=p\neq q$.

The contribution of the appropriate terms in the quadruple sum is

$$\sum_{\substack{n \\ n\neq q}} \sum_q |w_n|^2 w_n w_q^* \overline{e^{j(\delta_n - \delta_q)}} e^{j(d_n - d_q)u}$$

$$= \left|\overline{e^{j\delta}}\right|^2 \sum_{\substack{n \\ n\neq q}} \sum_q |w_n|^2 w_n w_q^* e^{j(d_n - d_q)u}$$

$$= \left|\overline{e^{j\delta}}\right|^2 \left[\sum_n \sum_q |w_n|^2 w_n w_q^* e^{j(d_n - d_q)u} - \sum_n |w_n|^4\right]$$

$$= \left|\overline{e^{j\delta}}\right|^2 \left[G_o(u) F_o^*(u) - \sum_n |w_n|^4\right]$$

where $G_o(u)$ is the pattern corresponding to the weights $|w_n|^2 w_n$; that is,

$$G_o(u) = \sum_{n=1}^{N} |w_n|^2 w_n e^{jd_n u} \tag{9}$$

(2b) $n=m=q\neq p$

Here the contribution of the quadruple sum is

$$\sum_{\substack{n \\ n\neq p}} \sum_p |w_n|^2 w_n^* w_p \overline{e^{j(\delta_p - \delta_n)}} e^{j(d_p - d_n)u}$$

$$= \left|\overline{e^{j\delta}}\right|^2 \sum_{\substack{n \\ n\neq p}} \sum_p |w_n|^2 w_n^* w_p e^{j(d_p - d_n)u}$$

$$= \left|\overline{e^{j\delta}}\right|^2 \left[\sum_n \sum_p |w_n|^2 w_n^* w_p e^{j(d_p - d_n)u} - \sum_n |w_n|^4\right]$$

$$= \left|\overline{e^{j\delta}}\right|^2 \left[G_o^*(u) F_o(u) - \sum_n |w_n|^4\right].$$

(2c) $n=p=q\neq m$

By symmetry, the contribution of these terms is the same as that for (2a).

(2d) $m=p=q\neq n$

By symmetry, the contribution of these terms is the same as that for (2b).

(3a) $n=m,\ p=q,\ n\neq p$

The contribution of the appropriate terms in Eq. (8) is

$$\sum_{\substack{n \\ n\neq p}} \sum_{p} |w_n|^2 |w_p|^2 = \sum_{n} \sum_{p} |w_n|^2 |w_p|^2 - \sum_{n} |w_n|^4$$

$$= \left(\sum_{n} |w_n|^2 \right)^2 - \sum_{n} |w_n|^4 \ .$$

(3b) $n=p,\ m=q,\ n\neq m$

The contribution of these terms is

$$\sum_{\substack{n \\ n\neq m}} \sum_{m} w_n^2 \ w_m^{*2} \ e^{j2(\delta_n - \delta_m)} \ e^{j2(d_n - d_m)u}$$

$$= \left| e^{j2\delta} \right|^2 \sum_{\substack{n \\ n\neq m}} \sum_{m} w_n^2 \ w_m^{*2} \ e^{j2(d_n - d_m)u}$$

$$= \left| \overline{e^{j2\delta}} \right|^2 \left[\sum_{n} \sum_{m} w_n^2 \ w_m^{*2} \ e^{j2(d_n - d_m)u} - \sum_{n} |w_n|^4 \right]$$

$$= \left| \overline{e^{j2\delta}} \right|^2 \left[\left| \sum_{n} w_n^2 \ e^{j2d_n u} \right|^2 - \sum_{n} |w_n|^4 \right]$$

$$= \left| \overline{e^{j2\delta}} \right|^2 \left[\left| H_o(u) \right|^2 - \sum_{n} |w_n|^4 \right]$$

where

$$H_o(u) = \sum_{n=1}^{N} w_n^2 \ e^{j2d_n u} \ . \qquad (10)$$

Note that $H_o(u)$ is the pattern of an array whose weights are the squares of the given array and whose elements are spaced twice as far apart.

(3c) $n=q,\ m=p,\ n\neq m$

By symmetry, the contribution of these terms is the same as that for (3a).

(4a) $n=m,\ n\neq p,\ n\neq q,\ p\neq q$

The corresponding terms in Eq. (8) contribute

$$\sum_{\substack{n \\ n\neq p,\ n\neq q}} \sum_{\substack{p \\ p\neq q}} \sum_{q} |w_n|^2 w_p w_q^* \overline{e^{j(\delta_p - \delta_q)}}\, e^{j(d_p - d_q)u}$$

$$= \left|\overline{e^{j\delta}}\right|^2 \sum_{\substack{n \\ n\neq p,\ n\neq q}} \sum_{\substack{p \\ p\neq q}} \sum_{q} |w_n|^2 w_p w_q^* e^{j(d_p - d_q)u}$$

$$= \left|\overline{e^{j\delta}}\right|^2 \left[\sum_{\substack{n \\ n\neq p,\ n\neq q}} \sum_{p} \sum_{q} |w_n|^2 w_p w_q^* e^{j(d_p - d_q)u} - \sum_{\substack{n \\ n\neq p}} \sum_{p} |w_n|^2 |w_p|^2\right]$$

$$= \left|\overline{e^{j\delta}}\right|^2 \Bigg[\sum_{n} \sum_{p} \sum_{q} |w_n|^2 w_p w_q^* e^{j(d_p - d_q)u}$$

$$- \sum_{\substack{n \\ n\neq q}} \sum_{q} |w_n|^2 w_n w_q^* e^{j(d_n - d_q)u} - \sum_{\substack{n \\ n\neq p}} \sum_{p} |w_n|^2 w_p w_n^* e^{j(d_p - d_n)} - \sum_{n} |w_n|^4$$

$$- \sum_{n} \sum_{p} |w_n|^2 |w_p|^2 + \sum_{n} |w_n|^4\Bigg]$$

$$= \left|\overline{e^{j\delta}}\right|^2 \Bigg[F_o(u)\, F_o^*(u) \sum_{n} |w_n|^2 - \sum_{n} \sum_{q} |w_n|^2 w_n w_q^* e^{j(d_n - d_q)u}$$

$$+ \sum_{n} |w_n|^4 - \sum_{n} \sum_{p} |w_n|^2 w_p w_n^* e^{j(d_p - d_n)u} + \sum_{n} |w_n|^4$$

$$- \left(\sum_{n} |w_n|^2\right)^2\Bigg]$$

$$= \left|\overline{e^{j\delta}}\right|^2 \Bigg[|F_o(u)|^2 \sum_{n} |w_n|^2 - 2\,\mathrm{Re}\left[G_o(u)\, F_o^*(u)\right] - \left(\sum_{n} |w_n|^2\right)^2$$

$$+ 2 \sum_{n} |w_n|^4\Bigg]$$

with $G_o(u)$ given by Eq. (9).

(4b) $n=p,\ n\neq m,\ n\neq q,\ m\neq q$

The contribution of these terms is

$$\sum_{\substack{n \quad m \quad q\\ m\neq q\\ n\neq m,\ n\neq q}} w_n^2 w_m^* w_q^* \overline{e^{j(2\delta_n-\delta_m-\delta_q)}}\ e^{j(2d_n-d_m-d_q)u}$$

$$= \overline{e^{j2\delta}}\ (\overline{e^{-j\delta}})^2 \sum_{\substack{n \quad m \quad q\\ m\neq q\\ n\neq m,\ n\neq q}} w_n^2\ w_m^*\ w_q^*\ e^{j(2d_n-d_m-d_q)u}$$

$$= \overline{e^{j2\delta}}\ (\overline{e^{-j\delta}})^2 \left[\sum_{\substack{n \quad m \quad q\\ n\neq m,\ n\neq q}} w_n^2 w_m^* w_q^* e^{j(2d_n-d_m-d_q)u} - \sum_{\substack{n \quad m\\ n\neq m}} w_n^2\ w_n^{*2}\ e^{j2(d_n-d_m)u}\right]$$

$$= \overline{e^{j2\delta}}\ (\overline{e^{-j\delta}})^2 \left[\sum_n \sum_m \sum_q w_n^2\ w_m^*\ w_q^* e^{j(2d_n-d_m-d_q)u} - \sum_{\substack{n \quad q\\ n\neq q}} |w_n|^2 w_n w_q^*\ e^{j(d_n-d_q)u} - \sum_{\substack{n \quad m\\ n\neq m}} |w_n|^2\ w_n\ w_m^* e^{j(d_n-d_m)u} - \sum_n |w_n|^4 - \sum_n \sum_m w_n^2\ w_m^{*2}\ e^{j2(d_n-d_m)u} + \sum_n |w_n|^4\right]$$

$$= \overline{e^{j2\delta}}\ (\overline{e^{-j\delta}})^2 \left[H_o(u)\ F_o^{*2}(u) - \sum_n \sum_q |w_n|^2 w_n w_q^* e^{j(d_n-d_q)u} + \sum_n |w_n|^4 - \sum_n \sum_m |w_n^2|\ w_n\ w_m^* e^{j(d_n-d_m)u} + \sum_n |w_n|^4 - H_o(u)\ H_o^*(u)\right]$$

$$= \overline{e^{j2\delta}}\ (\overline{e^{-j\delta}})^2 \left[H_o(u)\ F_o^{*2}(u) - 2G_o(u)\ F_o^*(u) - |H_o(u)|^2 + 2\sum_n |w_n|^4\right]$$

where $G_o(u)$ is given by Eq. (9) and $H_o(u)$ by Eq. (10).

(4c) n=q, n≠m, n≠p, m≠p

By symmetry, the contribution of these terms equals that of (4a).

(4d) m=p, m≠n, m≠q, n≠q

By symmetry, the contribution of these terms equals that of (4a).

(4e) m=q, m≠n, m≠p, n≠p

By symmetry, the contribution of these terms equals the complex conjugate of that of (4b).

(4f) p=q, p≠n, p≠m, n≠m

By symmetry, the contribution of these terms equals that of (4a). Hence of the six cases for which exactly two indices are equal, four are equivalent to (4a) and two are complex conjugates.

(5) no indices equal

These terms contribute

$$|\overline{e^{j\delta}}|^4 \underset{\text{no indices equal}}{\sum_n \sum_m \sum_p \sum_q} w_n w_m^* w_p w_q^* e^{j(d_n - d_m + d_p - d_q)u}$$

$$= |\overline{e^{j\delta}}|^4 \Bigg\{ \sum_n \sum_m \sum_p \sum_q w_n w_m^* w_p w_q^* e^{j(d_n - d_m + d_p - d_q)u}$$

$$- \sum_n |w_n|^4 - 2\left[G_o(u) F_o^*(u) - \sum_n |w_n|^4\right] - 2\left[G_o^*(u) F_o(u) - \sum_n |w_n|^4\right]$$

$$- 2\left[\left(\sum_n |w_n|^2\right)^2 - \sum_n |w_n|^4\right] - \left[|H_o(u)|^2 - \sum_n |w_n|^4\right]$$

$$- 4\left[|F_o(u)|^2 \sum_n |w_n|^2 - 2\,\mathrm{Re}\left(G_o(u) F_o^*(u)\right) - \left(\sum_n |w_n|^2\right)^2 \right.$$

$$\left. + 2 \sum_n |w_n|^4\right]$$

$$- 2\,\mathrm{Re}\left[H_o(u) F_o^{*2}(u) - 2 G_o(u) F_o^*(u) - |H_o(u)|^2 + 2 \sum_n |w_n|^4\right]\Bigg\}$$

$$= |\overline{e^{j\delta}}|^4 \left[|F_o(u)|^4 - 4|F_o(u)|^2 \sum_n |w_n|^2 + 8\,\mathrm{Re}\left[G_o(u) F_o^*(u)\right] \right.$$

$$\left. + |H_o(u)|^2 - 2\,\mathrm{Re}\left[H_o(u) F_o^{*2}(u)\right] + 2\left(\sum_n |w_n|^2\right)^2 - 6 \sum_n |w_n|^4\right] .$$

To obtain the first equality we have simply pooled the results of the preceeding cases.

The average of the squared power, $\overline{|F(u)|^4}$, is equal to the sum of the contributions of the separate cases we have analyzed. Equation (7) shows that to calculate the variance of the power we must subtract the square of the average power from the average of the squared power. From Eq. (4) the square of the average power is given by

$$\left(\overline{|F(u)|^2}\right)^2 = \left|\overline{e^{j\delta}}\right|^4 |F_o(u)|^4 + 2\left|\overline{e^{j\delta}}\right|^2 \left(1 - \left|\overline{e^{j\delta}}\right|^2\right) |F_o(u)|^2 \sum_n |w_n|^2$$

$$+ \left(1 - 2\left|\overline{e^{j\delta}}\right|^2 + \left|\overline{e^{j\delta}}\right|^4\right)\left(\sum_n |w_n|^2\right)^2 . \tag{11}$$

Subtracting Eq. (11) from $\overline{|F(u)|^4}$ and combining terms, the variance of the power is found to be

$$\begin{aligned}
\mathrm{Var}\left(|F(u)|^2\right) = {} & 2\left|\overline{e^{j\delta}}\right|^2 \left(1 - \left|\overline{e^{j\delta}}\right|^2\right)\left(\sum_n |w_n|^2\right) |F_o(u)|^2 \\
& - 4\left|\overline{e^{j\delta}}\right|^2 \left(1 + \left|\overline{e^{j2\delta}}\right| - 2\left|\overline{e^{j\delta}}\right|^2\right) \mathrm{Re}\left[G_o(u)\, F_o^*(u)\right] \\
& - 2\left|\overline{e^{j\delta}}\right|^2 \left(\left|\overline{e^{j\delta}}\right|^2 - \left|\overline{e^{j2\delta}}\right|\right) \mathrm{Re}\left[H_o(u)\, F_o^{*2}(u)\right] \\
& + \left(\left|\overline{e^{j\delta}}\right|^2 - \left|\overline{e^{j2\delta}}\right|\right)^2 |H_o(u)|^2 \\
& + \left(1 - \left|\overline{e^{j\delta}}\right|^2\right)^2 \left(\sum_n |w_n|^2\right)^2 \\
& - \left(1 - 4\left|\overline{e^{j\delta}}\right|^2 + \left|\overline{e^{j2\delta}}\right|^2 - 4\left|\overline{e^{j\delta}}\right|^2 \left|\overline{e^{j2\delta}}\right| + 6\left|\overline{e^{j\delta}}\right|^4\right)\left(\sum_n |w_n|^4\right)
\end{aligned}$$

We have shown above, in Eq. (3), that for phase errors uniformly distributed in the interval $[-\Delta, \Delta]$, $\Delta = \pi/2^{N_{bit}}$, $\overline{e^{j\delta}} = \frac{\sin\Delta}{\Delta}$. Similarly $\overline{e^{j2\delta}} = \frac{\sin 2\Delta}{2\Delta}$ and so

$$\mathrm{Var}\,(|F(u)|^2) = 2\left(\frac{\sin\Delta}{\Delta}\right)^2 \left[1 - \left(\frac{\sin\Delta}{\Delta}\right)^2\right]\left(\sum_n |w_n|^2\right) \quad |F_o(u)|^2$$

$$- 4\left(\frac{\sin\Delta}{\Delta}\right)^2 \left[1 + \frac{\sin 2\Delta}{2\Delta} - 2\left(\frac{\sin\Delta}{\Delta}\right)^2\right] \mathrm{Re}\left[G_o(u)\, F_o^*(u)\right]$$

$$- 2\left(\frac{\sin\Delta}{\Delta}\right)^2 \left[\left(\frac{\sin\Delta}{\Delta}\right)^2 - \frac{\sin 2\Delta}{2\Delta}\right] \mathrm{Re}\left[H_o(u)\, F_o^{*2}(u)\right]$$

$$+ \left[\left(\frac{\sin\Delta}{\Delta}\right)^2 - \frac{\sin 2\Delta}{2\Delta}\right]^2 |H_o(u)|^2$$

$$+ \left[1 - \left(\frac{\sin\Delta}{\Delta}\right)^2\right]^2 \left(\sum_n |w_n|^2\right)^2$$

$$- \left[1 - 4\left(\frac{\sin\Delta}{\Delta}\right)^2 + \left(\frac{\sin 2\Delta}{2\Delta}\right)^2 - 4\left(\frac{\sin\Delta}{\Delta}\right)^2 \frac{\sin 2\Delta}{2\Delta}\right.$$

$$\left. + 6\left(\frac{\sin\Delta}{\Delta}\right)^4\right] \left(\sum_n |w_n|^4\right) \tag{12}$$

where $G_o(u)$ and $H_o(u)$ are given by Eqs. (9) and (10) respectively. As above for the average power pattern, the variance of the power is the superposition of terms that are direction dependent and terms that are direction independent (that is, omnidirectional). For $\Delta = 0$ (that is, no phase errors), $\frac{\sin\Delta}{\Delta}$ and $\frac{\sin 2\Delta}{2\Delta}$ are both unity and Eq. (12) reduces to zero as required. In applications to phase shifters with many bits we can approximate the combinations of $\frac{\sin\Delta}{\Delta}$ and $\frac{\sin 2\Delta}{2\Delta}$ that appear in Eq. (12) by their small angle forms thus obtaining after some manipulation

$$\mathrm{Var}\left(|F(u)|\right)^2 \approx$$

$$\frac{2\Delta^2}{3}\left(1 - \frac{7}{15}\Delta^2\right)\left(\sum_n |w_n|^2\right) |F_o(u)|^2 - \frac{8}{45}\Delta^4\, \mathrm{Re}\,[G_o(u)\, F_o^*(u)]$$

$$- \frac{2\Delta^2}{3}\left(1 - \frac{3}{5}\Delta^2\right) \mathrm{Re}\,[H_o(u)\, F_o^{*2}(u)] + \frac{\Delta^4}{9}|H_o(u)|^2$$

$$+ \frac{\Delta^4}{9}\left(\sum_n |w_n|^2\right)^2 - \frac{2}{15}\Delta^4\left(\sum_n |w_n|^4\right), \quad N_{bit} > 4\,. \tag{13}$$

At a null of the error-free pattern, the variance of the power is given by

$$\mathrm{Var}\left(|F(u)|^2\right) \overset{F_o(u)=0}{=}$$

$$\left[\left(\frac{\sin\Delta}{\Delta}\right)^2 - \frac{\sin 2\Delta}{2\Delta}\right]^2 |H_o(u)|^2 + \left[1 - \left(\frac{\sin\Delta}{\Delta}\right)^2\right]^2 \left(\sum_n |w_n|^2\right)^2$$

$$- \left[1 - 4\left(\frac{\sin\Delta}{\Delta}\right)^2 + \left(\frac{\sin 2\Delta}{2\Delta}\right)^2 - 4\left(\frac{\sin\Delta}{\Delta}\right)^2 \frac{\sin 2\Delta}{2\Delta}\right.$$

$$\left. + 6\left(\frac{\sin\Delta}{\Delta}\right)^4\right] \sum_n |w_n|^4$$

$$\overset{N_{bit} > 4}{\approx} \frac{\Delta^4}{9}\left[|H_o(u)|^2 + \left(\sum_n |w_n|^2\right)^2 - \frac{6}{5}\left(\sum_n |w_n|^4\right)\right]. \qquad (14)$$

It is of interest to estimate the relative contribution of the three terms within the square brackets on the left hand side of Eq. (14). For phases of the error-free weights varying linearly over the array we can write $w_n = a_n e^{-j d_n u_s}$ and so from Eq. (10)

$$H_o(u) = \sum_n a_n^2 e^{j 2 d_n (u - u_s)}.$$

Hence $H_o(u)$ has a maximum magnitude of $\sum a_n^2$ at those directions for which $u = u_s \pm m\pi$ or equivalently

$$kd \sin\theta = kd \sin\theta_s \pm m\pi$$

or

$$\sin\theta = \sin\theta_s \pm m\frac{\lambda}{2d}.$$

If, for example, $d/\lambda = 1$ and $\theta_s = 0$ then in the visible region $H_o(u) = \sum a_n^2$ when $\sin\theta = 0, \pm 1/2$ or $\theta = 0°, \pm 30°$. As u moves away from these points, $H_o(u)$ falls off rapidly, especially for arrays with a strong taper of the amplitudes of the error-free weights, and so $|H_o(u)|^2$ can be neglected compared to $\left(\sum_n |w_n|^2\right)^2$.

As far as the relative contribution of the terms $\left(\sum |w_n|^2\right)^2$ and $\frac{6}{5} \sum |w_n|^4$ is concerned, applying Cauchy's inequality (Ref. 4) we obtain

$$\sum_n |w_n|^4 = \sum_n \left[\left(|w_n|^2\right)^2\right] \leq \left(\sum_n |w_n|^2\right)^2$$

and applying Chebyshev's inequality (Ref. 5) we obtain

$$N \sum_{n=1}^{N} |w_n|^4 = N \sum_{n=1}^{N} \left[\left(|w_n|^2\right)^2\right] \geq \left(\sum_n |w_n|^2\right)^2$$

so that

$$\frac{6}{5N} \leq \frac{\frac{6}{5} \sum_n |w_n|^4}{\left(\sum_n |w_n|^2\right)^2} \leq \frac{6}{5} . \tag{15}$$

Note that the left "≤" in Eq. (15) becomes equality for the case of a uniform amplitude array, and that the right "≤" becomes equality for the case of an array with all amplitudes except for one equal to zero. Furthermore it can be shown (see Appendix A) that for any amplitude taper describable by a polynominal in n, the

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4. Cauchy's Inequality:

$$\left[\sum_{k=1}^{n} a_k b_k\right]^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \qquad \text{(equality for } a_k = c\, b_k \text{, } c \text{ constant).}$$

See Abramowitz, M. (1972) Elementary analytical methods, Chap. 3 of Handbook of Mathematical Functions, M. Abramowitz and I. Stegun, (Eds.), Dover, N.Y.

5. Chebyshev's Inequality:

$$\text{If } a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n$$

$$b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_n$$

$$n \sum_{k=1}^{n} a_k b_k \geq \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

See Abramowitz, M.[4].

ratio of $\sum |w_n|^4$ to $\left(\sum |w_n|^2\right)^2$ is proportional to 1/N as N becomes large. Hence for large arrays and any usual amplitude taper, $\frac{6}{5} \sum |w_n|^4$ is of the order of 1/N compared with $\left(\sum |w_n|^2\right)^2$ and can be neglected.

It is often useful to express the variance of a quantity relative to the square of the mean of the quantity, thus giving a measure of variance divorced from the actual units in which the quantity is measured. [The ratio of the standard deviation (that is, the square root of the variance) to the mean is known as the coefficient of variation.] If Eq. (14) is divided by the square of Eq. (6b) we obtain

$$\frac{\mathrm{Var}\left(|F(u)|^2\right)}{\left[\overline{|F(u)|^2}\right]^2} = 1 + \frac{|H_o(u)|^2 - \frac{6}{5} \sum_n |w_n|^4}{\left(\sum_n |w_n|^2\right)^2} \,. \tag{16}$$

Note that, to within the restriction that Δ is small enough for $(\sin \Delta)/\Delta$ and $(\sin 2\Delta)/(2\Delta)$ to be well approximated by their small angle forms, the ratio of the variance of the power to the square of the mean power at a direction for which the error-free pattern has a null is independent of the magnitude of the phase errors. In accordance with the discussion of the previous paragraph, if u is not in the vicinity of those points for which $H_o(u)$ has its maxima, and if the number of elements in the array is large, the second term on the right hand side of Eq. (16) can be neglected compared with one and so the variance of the power is approximately equal to the square of the mean of the power. Equivalently, the standard deviation of the power is approximately equal to the mean of the power.

2.3 Probability Distribution of the Power

Having derived expressions for the mean and variance of the power, we now obtain the probability distribution of the amplitude of the field in a given direction of the pattern. By a simple transformation, we can then also obtain the probability distribution of the power in a given direction. Our procedure is essentially that of Allen[2] and Beckmann.[6] We first resolve the field into its real and imaginary components, each of which is given by a sum of N terms (N = number of elements in the array) involving trigonometric functions of the random phase errors. Since the phase errors are assumed to be mutually independent, the terms of the sums are independent random variables. We assume that N is sufficiently large (and that

6. Beckmann, P. (1967) Probability in Communication Engineering, Harcourt, Brace and World, Inc., New York, pp. 59-63.

no one term dominates the sum) for the Central Limit Theorem to apply, and so infer that the real and imaginary parts of the field are Gaussian distributed. It is straightforward to calculate the mean, variance, and covariance of the real and imaginary field components. We then transform from cartesian to polar coordinates and obtain the probability distribution of the amplitude of the field by integrating the joint probability distribution of the amplitude and phase of the field with respect to the phase.

Referring to Eq. (1), let the error-free weights be represented by $a_n e^{j\varphi_n}$. Then

$$F(u) = \sum_n a_n e^{j\varphi_n} e^{j\delta_n} e^{j d_n u}$$

$$= \sum_n a_n e^{j\psi_n} e^{j\delta_n}$$

where

$$\psi_n = \varphi_n + d_n u \, .$$

Then

$$X \equiv \mathrm{Re}[F(u)] = \sum_n a_n \cos(\psi_n + \delta_n) \tag{17a}$$

$$= \sum_n a_n (\cos\psi_n \cos\delta_n - \sin\psi_n \sin\delta_n) \tag{17b}$$

and

$$Y \equiv \mathrm{Im}[F(u)] = \sum_n a_n \sin(\psi_n + \delta_n) \tag{18a}$$

$$= \sum_n a_n (\sin\psi_n \cos\delta_n + \cos\psi_n \sin\delta_n) \, . \tag{18b}$$

Since our interest is in large arrays, and since for such arrays no one element can be singled out as dominant, we can apply the Central Limit Theorem to assert that the real and imaginary parts of the field are random variables with a joint bivariate Gaussian probability distribution. The statistics of this distribution are completely determined by the means, variances, and covariance, so that our task is to calculate these quantities.

The phase errors, δ_n, are assumed to be identically and uniformly distributed in the interval $[-\pi/2^{N_{bit}}, \pi/2^{N_{bit}}]$. Hence

$$\overline{\sin\delta_n} = 0$$

$$\overline{\cos\delta_n} = \frac{\sin\Delta}{\Delta}, \quad \Delta = \frac{\pi}{2^{N_{bit}}} \tag{19}$$

and thus from Eqs. (17b) and (18b), the means of the real and imaginary parts of the field are

$$\begin{aligned}\overline{X} &= \frac{\sin\Delta}{\Delta} \sum_n a_n \cos\psi_n \\ &= \frac{\sin\Delta}{\Delta} \operatorname{Re}[F_o(u)] ,\end{aligned} \tag{20}$$

and

$$\begin{aligned}\overline{Y} &= \frac{\sin\Delta}{\Delta} \sum_n a_n \sin\psi_n \\ &= \frac{\sin\Delta}{\Delta} \operatorname{Im}[F_o(u)] .\end{aligned} \tag{21}$$

To calculate the variances of the real and imaginary parts of the field, σ_X^2 and σ_Y^2, we write Eqs. (17a) and (18a) in the form

$$X = \sum_n x_n, \quad x_n = a_n \cos(\psi_n + \delta_n)$$

$$Y = \sum_n y_n, \quad y_n = a_n \sin(\psi_n + \delta_n)$$

and use the theorem that the variance of a sum of independent random variables equals the sum of their variances[3] to give

$$\sigma_X^2 = \sum_n \sigma_{x_n}^2$$

$$\sigma_Y^2 = \sum_n \sigma_{y_n}^2 .$$

But

$$\sigma^2_{x_n} = \overline{x^2_n} - \bar{x}^2_n$$

$$\sigma^2_{y_n} = \overline{y^2_n} - \bar{y}^2_n$$

and

$$\bar{x}_n = \frac{\sin \Delta}{\Delta} a_n \cos \psi_n ,$$

$$\bar{x}^2_n = \left(\frac{\sin \Delta}{\Delta}\right)^2 a^2_n \cos^2 \psi_n$$

$$= \frac{1}{2} \left(\frac{\sin \Delta}{\Delta}\right)^2 a^2_n [1 + \cos(2\psi_n)]$$

$$\overline{x^2_n} = a^2_n \overline{\cos^2(\psi_n + \delta_n)}$$

$$= \frac{1}{2} a^2_n \left\{1 + \overline{\cos[2(\psi_n + \delta_n)]}\right\}$$

$$= \frac{1}{2} a^2_n \left[1 + \cos(2\psi_n) \overline{\cos(2\delta_n)} - \sin(2\psi_n) \overline{\sin(2\delta_n)}\right]$$

$$= \frac{1}{2} a^2_n \left[1 + \frac{\sin 2\Delta}{2\Delta} \cos(2\psi_n)\right] ,$$

$$\bar{y}_n = \frac{\sin \Delta}{\Delta} a_n \sin \psi_n$$

$$\bar{y}^2_n = \left(\frac{\sin \Delta}{\Delta}\right)^2 a^2_n \sin^2 \psi_n$$

$$= \frac{1}{2} \left(\frac{\sin \Delta}{\Delta}\right)^2 a^2_n [1 - \cos(2\psi_n)] ,$$

$$\overline{y^2_n} = a^2_n \overline{\sin^2(\psi_n + \delta_n)}$$

$$= \frac{1}{2} a^2_n \left\{1 - \overline{\cos[2\psi_n + \delta_n)]}\right\}$$

$$= \frac{1}{2} a^2_n \left[1 - \frac{\sin 2\Delta}{2\Delta} \cos(2\psi_n)\right]$$

where we have made use of the trigonometric identities

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta) \tag{22}$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) .$$

Hence

$$\sigma^2_{x_n} = \frac{1}{2}\left[1 - \left(\frac{\sin \Delta}{\Delta}\right)^2\right] a_n^2 - \frac{1}{2}\left[\left(\frac{\sin \Delta}{\Delta}\right)^2 - \frac{\sin 2\Delta}{2\Delta}\right] a_n^2 \cos (2\psi_n)$$

$$\sigma^2_{y_n} = \frac{1}{2}\left[1 - \left(\frac{\sin \Delta}{\Delta}\right)^2\right] a_n^2 + \frac{1}{2}\left[\left(\frac{\sin \Delta}{\Delta}\right)^2 - \frac{\sin 2\Delta}{2\Delta}\right] a_n^2 \cos (2\psi_n)$$

and

$$\sigma^2_X = \frac{1}{2}\left[1 - \left(\frac{\sin \Delta}{\Delta}\right)^2\right] \sum_n a_n^2 - \frac{1}{2}\left[\left(\frac{\sin \Delta}{\Delta}\right)^2 - \frac{\sin 2\Delta}{2\Delta}\right] \sum_n a_n^2 \cos (2\psi_n) \tag{23}$$

$$\sigma^2_Y = \frac{1}{2}\left[1 - \left(\frac{\sin \Delta}{\Delta}\right)^2\right] \sum_n a_n^2 + \frac{1}{2}\left[\left(\frac{\sin \Delta}{\Delta}\right)^2 - \frac{\sin 2\Delta}{2\Delta}\right] \sum_n a_n^2 \cos (2\psi_n) . \tag{24}$$

The covariance of the real and imaginary parts of the field is defined by

$$\sigma_{XY} = \overline{XY} - \overline{X}\,\overline{Y} .$$

Now, from Eqs. (17b) and (18b),

$$\overline{XY} = \sum_n \sum_{\substack{m \\ n \neq m}} a_n a_m \overline{(\cos \psi_n \cos \delta_n - \sin \psi_n \sin \delta_n)(\sin \psi_m \cos \delta_m + \cos \psi_m \sin \delta_m)}$$

$$+ \frac{1}{2} \sum_n a_n^2 \overline{\sin [2(\psi_n + \delta_n)]} \tag{25}$$

where, as above, we have separated the double summation into the terms for which $n \neq m$ and hence δ_n and δ_m are independent, and the terms for which $n = m$. The double summation in Eq. (25) reduces to

$$\left(\frac{\sin\Delta}{\Delta}\right)^2 \sum_{n} \sum_{\substack{m \\ n \neq m}} a_n a_m \cos\psi_n \sin\psi_m$$

because of Eq. (19), and

$$\frac{1}{2}\sum_n a_n^2 \overline{\sin[2(\psi_n + \delta_n)]} = \frac{1}{2}\sum_n a_n^2 [\sin(2\psi_n)\,\overline{\cos(2\delta_n)} + \cos(2\psi_n)\,\overline{\sin(2\delta_n)}]$$

$$= \frac{1}{2}\frac{\sin 2\Delta}{2\Delta}\sum_n a_n^2 \sin(2\psi_n) .$$

Hence

$$\overline{XY} = \left(\frac{\sin\Delta}{\Delta}\right)^2 \left[\sum_n \sum_m a_n a_m \cos\psi_n \sin\psi_m - \sum_n a_n^2 \cos\psi_n \sin\psi_n\right]$$

$$+ \frac{1}{2}\frac{\sin 2\Delta}{2\Delta}\sum_n a_n^2 \sin(2\psi_n)$$

$$= -\frac{1}{2}\left[\left(\frac{\sin\Delta}{\Delta}\right)^2 - \frac{\sin 2\Delta}{2\Delta}\right]\sum_n a_n^2 \sin(2\psi_n)$$

$$+ \left(\frac{\sin\Delta}{\Delta}\right)^2 \mathrm{Re}\left[F_o(u)\right] I_m\left[F_o(u)\right]$$

and

$$\sigma_{XY} = -\frac{1}{2}\left[\left(\frac{\sin\Delta}{\Delta}\right)^2 - \frac{\sin 2\Delta}{2\Delta}\right]\sum_n a_n^2 \sin(2\psi_n) .$$

We have thus obtained expressions for all the quantities—means, variances, and covariance—needed to define the joint probability distribution of the real and imaginary parts of the field. Although it is possible to proceed to obtain the distribution of the amplitude of the error field for the general case of a complex error-free field,[6] for our purposes here it is sufficient to restrict attention to the case of a real error-free field. If $\mathrm{Im}[F_o(u)] = 0$, we see from Eq. (21) that $\overline{Y} = 0$. Furthermore, if the imaginary part of the field is zero, then the amplitudes of the array element weights must be even-symmetric and the phases of the weights odd-

symmetric with respect to the center of the array.[7] Hence the covariance of the real and imaginary parts of the field vanishes. We then have

$$\overline{X} = \frac{\sin \Delta}{\Delta} F_o(u) \tag{26}$$

$$\overline{Y} = 0 \tag{27}$$

$$\sigma_X^2 = \frac{1}{2}\left[1 - \left(\frac{\sin\Delta}{\Delta}\right)^2\right]\sum_n a_n^2 - \frac{1}{2}\left[\left(\frac{\sin\Delta}{\Delta}\right)^2 - \frac{\sin 2\Delta}{2\Delta}\right]\sum_n a_n^2 \cos(2\psi_n) \tag{28}$$

$$\sigma_Y^2 = \frac{1}{2}\left[1 - \left(\frac{\sin\Delta}{\Delta}\right)^2\right]\sum_n a_n^2 - \frac{1}{2}\left[\left(\frac{\sin\Delta}{\Delta}\right)^2 - \frac{\sin 2\Delta}{2\Delta}\right]\sum_n a_n^2 \cos(2\psi_n) \tag{29}$$

$$\sigma_{XY} = 0$$

and the joint probability distribution of the real and imaginary parts of the field is given by

$$p(X, Y) = \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left[-\frac{(X-\overline{X})^2}{2\sigma_X^{\;2}} - \frac{Y^2}{2\sigma_Y^{\;2}}\right]. \tag{30}$$

We now transform to polar coordinates letting

$$X = |F| \cos\theta$$

$$Y = |F| \sin\theta \tag{31}$$

$$dX\,dY = |F|\, d|F|\, d\theta$$

so that $|F|$ is the amplitude and θ the phase of the field. Note that since $|F|^2 = X^2 + Y^2$, the mean power is given by

$$\overline{|F|^2} = \overline{X^2} + \overline{Y^2}$$

$$= \overline{X}^2 + \overline{Y}^2 + \sigma_X^{\;2} + \sigma_Y^{\;2}$$

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7. Oppenheim, A.V., and Schafer, R.W. (1975) Digital Signal Processing, Prentice-Hall, New Jersey, pp. 24-26.

and hence from Eqs. (20), (21), (23), and (24), we have in general (that is, with no restriction to real error-free fields)

$$\overline{|F|^2} = \left(\frac{\sin\Delta}{\Delta}\right)^2 |F_o(u)|^2 + \left[1 - \left(\frac{\sin\Delta}{\Delta}\right)^2\right] \sum_n a_n^2$$

in agreement with Eq. (4) which gives the average power obtained more directly above. Substituting Eq. (31) into Eq. (30),

$$p(|F|, \theta) = \frac{|F|}{2\pi\,\sigma_X\sigma_Y} \exp\left[- \frac{(|F| \cos\theta - \overline{X})^2}{2\sigma_X^{\;2}} - \frac{(|F| \sin\theta)^2}{2\sigma_Y^{\;2}}\right]$$

$$= \frac{|F|}{2\pi\,\sigma_X\sigma_Y} \exp\left[- \frac{1}{2}\left(\frac{|F|^2\cos^2\theta - 2\overline{X}\,|F| \cos\theta + \overline{X}^2}{\sigma_X^{\;2}} + \frac{|F|^2 \sin^2\theta}{\sigma_Y^{\;2}}\right)\right].$$

Using the trigonometric identities of Eq. (22) we obtain

$$p(|F|, \theta) = \frac{|F|}{2\pi\,\sigma_X\sigma_Y} \exp\left[- \frac{1}{4}\left(\frac{1}{\sigma_X^{\;2}} + \frac{1}{\sigma_Y^{\;2}}\right) |F|^2 - \frac{\overline{X}^2}{2\sigma_X^{\;2}}\right]$$

$$\exp\left[- \frac{1}{4}\left(\frac{1}{\sigma_X^{\;2}} - \frac{1}{\sigma_Y^{\;2}}\right) |F|^2 \cos(2\theta) + \frac{\overline{X}\,|F| \cos\theta}{\sigma_X^{\;2}}\right]$$

$$= \frac{|F|}{2\pi\,\sigma_X\sigma_Y}\; e^{-D}\, e^{-P \cos(2\theta) + Q \cos\theta}$$

where

$$D = \frac{1}{4}\left(\frac{1}{\sigma_X^{\;2}} + \frac{1}{\sigma_Y^{\;2}}\right) |F|^2 + \frac{\overline{X}^2}{2\sigma_X^{\;2}} \tag{32a}$$

$$P = \frac{1}{4}\left(\frac{1}{\sigma_X^{\;2}} - \frac{1}{\sigma_Y^{\;2}}\right) |F|^2 \tag{32b}$$

$$Q = \frac{\overline{X}\,|F|}{\sigma_X^{\;2}} . \tag{32c}$$

The probability distribution of the field amplitude is then found by integrating $p(|F|, \theta)$ with respect to θ,

$$p(|F|) = \int_0^{2\pi} (|F|, \theta)\, d\theta$$

$$= 2\int_0^{\pi} p(|F|, \theta)\, d\theta$$

$$= \frac{|F|}{\pi\, \sigma_X \sigma_Y}\; e^{-D}\, I$$

where

$$I = \int_0^{\pi} e^{-P\cos(2\theta) + Q\cos\theta}\, d\theta\ .$$

Proceeding as in Reference 6 we obtain an expression for the integral I in terms of a series of modified Bessel functions. First, using the identities in Reference 8

$$e^{z\cos\theta} = \sum_{k=0}^{\infty} \epsilon_k I_k(z)\cos(k\theta), \quad \epsilon_k = \begin{cases} 1, & k = 0 \\ 2, & k \neq 0 \end{cases}$$

$$I_k(-z) = (-1)^k\, I_k(z)$$

we can write

$$e^{-P\cos(2\theta)} = \sum_{k=0}^{\infty} (-1)^k \epsilon_k I_k(P)\cos(2k\theta)$$

so that

$$I = \sum_{k=0}^{\infty} (-1)^k \epsilon_k I_k(P) \int_0^{\pi} e^{Q\cos\theta} \cos(2k\theta)\, d\theta\ .$$

Then, using the relation in Reference 8

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\, d\theta$$

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8. Olver, F.W.J. (1972) Bessel functions of integer order, Chap. 9 in Handbook of Mathematical Functions, M. Abramowitz and I. Stegun, Eds., Dover, N.Y., pp. 376-377.

we have

$$\int_0^{\pi} e^{Q\cos\theta} \cos(2k\theta)\, d\theta = \pi\, I_{2k}(Q)$$

so that

$$I = \pi \sum_{k=0}^{\infty} (-1)^k \epsilon_k I_k(P)\, I_{2k}(Q)$$

and the desired probability distribution of the field amplitude is

$$p(|F|) = \frac{|F|}{\sigma_X \sigma_Y} e^{-D} \sum_{k=0}^{\infty} (-1)^k \epsilon_k I_k(P)\, I_{2k}(Q) \tag{33}$$

with D, P, and Q given by Eqs. (32a-c). The distribution can be normalized by letting

$$B^2 = \frac{\bar{X}^2}{\sigma_X^{\ 2} + \sigma_Y^{\ 2}} \tag{34a}$$

$$|f| = \frac{|F|}{\sqrt{\sigma_X^{\ 2} + \sigma_Y^{\ 2}}} \tag{34b}$$

and

$$K = \frac{\sigma_Y}{\sigma_X} \tag{34c}$$

so that B^2 is the ratio of the constant component of the power to the average power in the fluctuating component of the power, and K is a measure of the asymmetry in the average proportion of the fluctuating component of the power associated with the real and imaginary parts of the field. Substituting Eq. (34) into Eq. (33) we obtain

$$p(|f|) = \frac{K^2+1}{K} |f| \exp\left[-\left(\frac{1+K^2}{2}\right)\left(B^2 + \frac{1+K^2}{2K^2}|f|^2\right)\right]$$
$$\times \sum_{k=0}^{\infty} (-1)^k I_k\left(\frac{K^4-1}{4K^2}|f|^2\right) I_{2k}\left[B(1+K^2)|f|\right]. \tag{35}$$

Curves of this distribution are given by Beckmann in References 9 and 10. The probability density function of the power can be obtained if desired from the probability density function of the field amplitude by the relations in Reference 6

$$p_{|F|^2}(|F|^2) = \frac{1}{2|F|}\, p_{|F|}(|F|) \tag{36}$$

$$p_{|f|^2}(|f|^2) = \frac{1}{2|f|}\, p_{|f|}(|f|)$$

using the notation $p_y(x)$ to mean the probability density of the random variable y as a function of x.

The combinations of $\frac{\sin \Delta}{\Delta}$ and $\frac{\sin 2\Delta}{\Delta}$ appearing in Eqs. (28) and (29) for the variance of the real and imaginary parts of the field can be replaced by their small angle forms for $N_{bit} > 4$ thus yielding

$$\sigma_X{}^2 \approx \frac{\Delta^2}{6}\left[\sum_n a_n^2 - \sum_n a_n^2 \cos 2(d_n u + \phi_n)\right] \tag{37}$$

$$\sigma_Y{}^2 \approx \frac{\Delta^2}{6}\left[\sum_n a_n^2 + \sum_n a_n^2 \cos 2(d_n u + \phi_n)\right]. \tag{38}$$

The second term within the square brackets is (for real fields) the function $H_o(u)$ encountered earlier [see Eq. (10)] in the course of obtaining an expression for the variance of the power. We showed above (see p. 19) that $H_o(u)$ can be neglected compared to $\sum a_n^2$ unless u is close to the points, midway between grating lobes, for which $|H_o(u)|$ takes on its maximum value. Then

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9. Beckmann, P., and Spizzichino, A. (1963) The Scattering of Electromagnetic Waves From Rough Surfaces, Pergamon Press, Macmillan, New York, Appendix F.

10. Beckmann, P. (1963) Statistical distribution of the amplitude and phase of a multiply scattered field, J. Res. Natl. Bur. Std., D. 66(No. 3):231-240.

$$\sigma_X{}^2 \approx \sigma_Y{}^2 \approx \frac{\Delta^2}{6} \sum_n a_n^2 \equiv \sigma^2 \tag{39}$$

and K = 1. Since $I_o(0) = 1$ and $I_k(0) = 0$ for $k \neq 0$, Eq. (35), which gives the probability density of the normalized field amplitude, reduces to

$$p(|f|) = 2\,|f| \exp\left[-(B^2 + |f|^2)\right] \; I_o\left(\frac{\sqrt{2}\,\overline{X}\,|f|}{\sigma}\right)$$

or

$$p(|F|) = \frac{|F|}{\sigma^2} \exp\left[-\frac{(\overline{X}^2 + |F|^2)}{2\sigma^2}\right] I_o\left(\frac{\overline{X}|F|}{\sigma^2}\right) \tag{40}$$

a distribution known as the Rice-Nakagami distribution and treated in slightly different forms in detail by Rice[11] and by Norton et al in Reference 12.*

11. Rice, S.O. (1954) Mathematical analysis of random noise, reprinted in Selected Papers on Noise and Stochastic Processes, N. Wax. (Eds.), Dover, New York, pp. 239-241.

12. Norton, K.A., et al (1955) The probability distribution of the amplitude of a constant vector plus a Rayleigh-distributed vector, Proc. IRE 43:1354-1361.

*Norton et al discuss the complement of the cumulative distribution function of the probability density function

$$p(r) = \frac{2}{k^2}\, r \exp\left[-(1 + r^2)/k^2\right] \; I_o\left(\frac{2r}{k^2}\right)$$

which is obtained from Eq. (40) via the correspondence

$$r = |F| - |\overline{X}| + 1, \quad k^2 = 2\sigma^2$$

Figure 4 of Norton et al contains plots of the complement of the cumulative distribution function of $\frac{r-1}{k}$ for various values of the parameter $K = 20 \log_{10}(k)$. In our notation this is equivalent to plotting the complement of the cumulative distribution function of $\frac{|F| - |\overline{X}|}{\sqrt{2}\sigma}$ for various values of $K = 10 \log_{10}(2\sigma^2)$. Rice discusses the probability density function

$$p(v) = v \exp\left(-\frac{v^2 + \alpha^2}{2}\right) I_o(\alpha v)$$

which can be obtained from Eq. (40) by letting

$$v = \frac{|F|}{\sigma}, \quad \alpha = \frac{|\overline{X}|}{\sigma}$$

He gives plots of both p(v) and its cumulative distribution function.

For our purposes it is somewhat more convenient to follow Rice's normalization

$$v = \frac{|F|}{\sigma}, \quad \alpha = \frac{|\overline{X}|}{\sigma}. \tag{41}$$

For small values of the parameter α the cumulative distribution function

$$C(v) = \int_0^v v \exp\left(-\frac{v^2 + \alpha^2}{2}\right) I_0(\alpha v)\, dv \tag{42}$$

can be obtained by numerical integration. For large values of α Rice obtains an asymptotic expansion for C(v) in inverse powers of α by first replacing $I_0(\alpha v)$ by its large argument asymptotic form, given in Reference 8,

$$I_0(\alpha v) \sim \frac{e^{\alpha v}}{\sqrt{2\pi\alpha v}} (1 + \alpha v)$$

so that

$$p(v) \sim \frac{1}{2\pi}\left(\frac{v}{\alpha}\right)^{1/2}\left(1 + \frac{1}{\alpha v}\right) \exp\left[-\frac{(v-\alpha)^2}{2}\right]$$

and then expanding $\left(\frac{v}{\alpha}\right)^{1/2}\left(1+\frac{1}{\alpha v}\right)$ in a series of powers of $\frac{v-\alpha}{\alpha}$ and integrating term by term. The resulting expansion through terms α^{-3} is

$$C(v) \sim \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{v-\alpha}{\sqrt{2}}\right)$$
$$- \frac{1}{2\alpha\sqrt{2\pi}}\left[1 - \frac{v-\alpha}{4\alpha} + \frac{1+(v-\alpha)^2}{8\alpha^2}\right] \exp\left[-\frac{(v-\alpha)^2}{2}\right]. \tag{43}$$

In Table 1 we compare values of the Rician cumulative distribution function obtained by performing the integration in Eq. (42) numerically (using the IMSL routine DCADRE) with values obtained using the asymptotic form of Eq. (43). It is seen that the asymptotic form gives an excellent fit to the exact values for $\alpha > 3$.

If we further specialize our results and consider the probability distribution of the amplitude at a direction for which the error-free pattern has a null, then $\overline{X}$ given by Eq. (26) equals zero and $p(|F|)$ in Eq. (40) reduces to the Rayleigh distribution

$$p(|F|) = \frac{|F|}{\sigma^2} \exp\left(-\frac{|F|^2}{2\sigma^2}\right).$$

Table 1. Values of the Rician Cumulative Distribution Function Calculated With the Asymptotic Expression, Compared With Values Obtained by Numerical Integration

	$v - \alpha$									
	-2		-1		0		1		2	
α	Asymp.	Integ	Asymp.	Integ.	Asymp.	Integ.	Asymp.	Integ.	Asymp.	Integ.
8	0.1913(-1)	0.1912(-1)	0.1430	0.1430	0.4750	0.4750	0.8266	0.8266	0.9741	0.9741
6	0.1780(-1)	0.1777(-1)	0.1375	0.1375	0.4666	0.4666	0.8219	0.8219	0.9730	0.9731
5	0.1668(-1)	0.1662(-1)	0.1330	0.1330	0.4599	0.4599	0.8181	0.8181	0.9723	0.9723
4	0.1489(-1)	0.1472(-1)	0.1260	0.1259	0.4497	0.4497	0.8125	0.8126	0.9711	0.9711
3	0.1163(-1)	0.1083(-1)	0.1138	0.1133	0.4326	0.4325	0.8033	0.8035	0.9691	0.9693

According to Eq. (36), the probability density function of the power is then the exponential distribution

$$p_{|F|^2}(|F|^2) = \frac{1}{2\sigma^2} \exp\left(-\frac{|F|^2}{2\sigma^2}\right). \tag{44}$$

The mean power is given by

$$\overline{|F|^2} = \frac{1}{2\sigma^2}\int_0^\infty |F|^2 e^{-\frac{|F|^2}{2\sigma^2}} d(|F|^2) = 2\sigma^2 \approx \frac{\Delta^2}{3} \sum_n a_n^2 \tag{45}$$

in agreement with Eq. (6b). The variance of the power is obtained from

$$\frac{1}{2\sigma^2}\int_0^\infty \left(|F|^2 - \overline{|F|^2}\right)^2 e^{-\frac{|F|^2}{2\sigma^2}} d(|F|^2)$$

$$= \frac{1}{2\sigma^2}\int_0^\infty y^2 e^{-\frac{y}{2\sigma^2}} dy - \overline{|F|^2}^{\,2}$$

$$= 2\,(2\sigma^2)^2 - (2\sigma^2)^2$$

$$= (2\sigma^2)^2 \approx \frac{\Delta^4}{9}\left(\sum_n a_n^2\right)^2 . \tag{46}$$

Comparing Eq. (46) with the expression in Eq. (14) for the variance of the power at a null of the error-free pattern obtained earlier and neglecting $H_0(u)$ in Eq. (14) to be consistent with the assumption made in obtaining Eqs. (39) and (40), we see that Eq. (14) contains the extra term $\frac{6}{5}\sum |w_n|^4$. This apparent discrepancy is explained by the fact that our derivation of the probability distribution of the amplitude and power is based on the application of the Central Limit Theorem and so is an approximation to the actual probability distribution. This approximate distribution approaches the actual distribution as the number of elements in the array, N, becomes large, provided no one amplitude of the element weights can be singled out as dominant. Our derivation of the expression for the variance of the power [Eq. (14)] involved no approximation and so cannot be expected to be identical to the expression for the variance obtained by applying the Central Limit Theorem except under the same conditions that the Central Limit Theorem applies. But we have seen earlier (see p. 20) that $\sum |w_n|^4$ is of the order of $1/N$ compared with $(\sum |w_n|^2)^2$ for any amplitude taper described by a polynomial (that is, any usual amplitude taper) and indeed for such tapers no one amplitude is dominant. Hence $\frac{6}{5}\sum |w_n|^4$ can be neglected compared to $(\sum |w_n|^2)^2$ under exactly the conditions

for which the Central Limit Theorem applies, thus yielding agreement between Eqs. (46) and (14). Note that the approximate equality of the variance of the power and the square of the mean power demonstrated above in connection with Eq. (16) approaches actual equality as the number of elements in the array increases and the probability distribution of the power approaches the exponential distribution for which, as we have seen, the variance is exactly equal to the square of the mean.

When nulls are placed in array patterns adaptively, the presence of multiple sources of interference can make it important to have nulls at each of several different directions. Accordingly it is of interest to consider the effect of quantized phase errors on the pattern at several directions simultaneously, for each of which the error-free pattern has a theoretically perfect null. It is then natural to examine the distribution of the least deep null among the several different directions, since the least deep null may serve as a limiting factor in determining the performance of the array. Now, the probability that the least deep null is less than or equal to a given level is equal to the probability that the pattern at all the specified directions is less than or equal to that given level. If we make the simplifying assumption that, for a given distribution of phase errors across the array, the null depths at any two locations for which the error-free pattern has a null are independent of each other, then we can obtain the desired probability simply by multiplying together the probabilities that each null is less than or equal to the given level. But from the discussion above [see Eq. (44)], the probability distribution of the power at any location for which the error free pattern has a null is given by

$$C(|F|^2) = \frac{1}{2\sigma^2}\int_0^{|F|^2} e^{-\frac{|F|^2}{2\sigma^2}}\, d(|F|^2) = 1 - e^{-\frac{|F|^2}{2\sigma^2}} . \tag{47}$$

Hence the cumulative probability distribution of the least deep null among M locations for which the error-free pattern has perfect nulls is given by

$$C_M(|F|^2) = \left(1 - e^{-\frac{|F|^2}{2\sigma^2}}\right)^M . \tag{48}$$

The corresponding probability density function is

$$\begin{aligned} p_M(|F|^2) &= \frac{d}{d(|F|^2)}\, C_M(|F|^2) \\ &= \frac{M}{2\sigma^2}\left(1 - e^{-\frac{|F|^2}{2\sigma^2}}\right)^{M-1} e^{-\frac{|F|^2}{2\sigma^2}} \end{aligned}$$

or

$$p_M(\zeta) = M\,(1 - e^{-\zeta})^{M-1}\, e^{-\zeta}$$

where we have let

$$\zeta = \frac{|F|^2}{2\sigma^2}.$$

The mean and variance of this distribution can be calculated analytically using the formula given in Reference 13

$$\int_0^\infty \zeta^{\nu-1}\, e^{-\zeta}\,(1 - e^{-\zeta})^{M-1}\, d\zeta = (-1)^{M-1}\,\Gamma(\nu) \sum_{k=0}^{M-1} (-1)^k \binom{M-1}{k} \frac{1}{(M-k)^\nu}\,.$$

To obtain the mean we let $\nu = 2$ so that

$$\bar{\zeta} = (-1)^{n-1}\, M \sum_{k=0}^{M-1} (-1)^k \binom{M-1}{k} \frac{1}{(M-k)^2}$$

and

$$\overline{|F|^2} = 2\sigma^2\,\bar{\zeta}\,.$$

To obtain the variance we let $\nu = 3$. Hence

$$\mathrm{Var}(\zeta) = M \int_0^\infty \zeta^2\,(1 - e^{-\zeta})^{M-1}\, e^{-\zeta}\, d\zeta - \bar{\zeta}^2$$

$$= (-1)^{M-1}\,(2M) \sum_{k=0}^{\infty} (-1)^k \binom{M-1}{k} \frac{1}{(M-k)^3} - \bar{\zeta}^2$$

and

$$\mathrm{Var}(|F|^2) = (2\sigma^2)^2\,\mathrm{Var}(\zeta)\,.$$

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13. Gradshteyn, I.S., and Ryzhik, I.M. (1980) Table of Integrals, Series, and Products, Academic Press, pp. 317, 333.

For the first few values of M we have the results in Table 2.

Table 2. Mean and Variance of the Probability Density Function of the Least Deep Null Among M Locations

M	ζ	$\mathrm{Var}(\zeta)$	
1	1	1	$\|F\|^2 = 2\sigma^2 \zeta$
2	3/2	5/4	$\mathrm{Var}(\|F\|^2) = (2\sigma^2)^2 \; \mathrm{Var}(\zeta)$
3	11/6	49/36	$2\sigma^2 = \frac{\Delta^2}{3} \sum a_n^2$
4	25/12	205/144	

Note that for M = 1, the values for the mean and variance of the least deep null agree with those found above [compare with Eqs. (45) and (46)] for a single null.

The final special case we shall consider here is that of the probability distribution of the amplitude and power at directions midway between grating lobes. For a linear phase variation, $\phi_n = -d_n u_s$, of the element weights across the array, these directions are given by

$$u = u_s \pm m\pi .$$

Since the focus of this report is the influence of phase errors on sidelobe levels we will restrict our attention to the case of m odd. (If m is even there is a grating lobe at the location u. The assumption that m is odd is not immediately needed here but is required in demonstrating agreement between the exact expression for the variance of the power derived earlier and the expression derived below from the probability distribution of the amplitude and power for this special case.) For these directions it is easily verified that

$$\sum_n a_n^2 \cos\left[2(\phi_n + d_n u)\right] = \pm \sum_n a_n^2 \tag{49}$$

so that from the approximations (37) and (38), either

$$\sigma_X^2 \approx 0, \quad \sigma_Y^2 \approx \frac{\Delta^2}{3} \sum_n a_n^2 \tag{50}$$

or

$$\sigma_Y^2 \approx 0, \quad \sigma_X^2 \approx \frac{\Delta^2}{3} \sum_n a_n^2 \tag{51}$$

according as the plus or minus sign respectively is taken in Eq. (49). Rather than attempting to specialize the general expressions we have derived above for the probability distribution of the amplitude and power to cover these cases, it is much simpler to derive the desired distributions starting with the variances of the real and imaginary parts of the field, Eqs. (50) or (51). (The means of the real and imaginary parts of the field are, from Eqs. (26) and (27), given by $\frac{\sin \Delta}{\Delta} F_o(u)$ and 0 respectively.)

For the first case, Eq. (50), the amplitude of the field is given by $|F| = (\overline{X}^2 + Y^2)^{1/2}$ so that $Y = \pm (|F|^2 - \overline{X}^2)^{1/2}$. Using the formula for the probability distribution of a function of a random variable[7] and the fact that Y is Gaussian distributed we have immediately

$$p(|F|) = \frac{1}{\sigma_Y} \left(\frac{2}{\pi} \right)^{1/2} \frac{|F|}{(|F|^2 - \overline{X}^2)^{1/2}} \, e^{-\frac{|F|^2 - \overline{X}^2}{2\sigma_Y^2}}, \quad |F| \geq \overline{X} .$$

The probability density of the power is, from Eq. (36), given by

$$p_{|F|^2}(|F|^2) = \frac{1}{(2\pi)^{1/2} \sigma_Y} \frac{1}{(|F|^2 - \overline{X}^2)^{1/2}} \, e^{-\frac{|F|^2 - \overline{X}^2}{2\sigma_Y^2}}, \quad |F|^2 \geq \overline{X}^2$$

and a short calculation then gives the cumulative distribution function of the power,

$$C(|F|^2) = \operatorname{erf} \left[\left(\frac{|F|^2 - \overline{X}^2}{2\sigma_Y^2} \right)^{1/2} \right], \quad |F|^2 \geq \overline{X}^2 \tag{52}$$

where erf(z) is the error function,

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt .$$

The mean power is given by

$$\overline{|F|^2} = \frac{1}{(2\pi)^{1/2} \sigma_Y} \int_{\overline{X}^2}^{\infty} \frac{|F|^2}{(|F|^2 - \overline{X}^2)^{1/2}} \, e^{-\frac{(|F|^2 - \overline{X}^2)}{2\sigma_Y^2}} \, d(|F|^2) .$$

Making the substitution $v = |F|^2 - \overline{X}^2$, $dv = d(|F|^2)$, and using the formulas given in Reference 13

$$\int_0^{\infty} x^{1/2} e^{-\mu x} = \left(\frac{\pi}{\mu}\right)^{1/2}, \quad \int_0^{\infty} x^{n-\frac{1}{2}} e^{-\mu x} dx = \pi^{1/2} \frac{1}{2} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{2n-1}{2} \mu^{-n-\frac{1}{2}} \tag{53}$$

we find

$$\overline{|F|^2} = \overline{X}^2 + \sigma_Y^2 \approx \left(1 - \frac{\Delta^2}{3}\right) F_o^2(u) + \frac{\Delta^2}{3} \sum_n a_n^2 \tag{54}$$

in agreement with Eq. (5).

The variance of the power is given by

$$\frac{1}{(2\pi)^{1/2}\sigma_Y} \int_{\overline{X}^2}^{\infty} \frac{|F|^4}{(|F|^2 - \overline{X}^2)^{1/2}} e^{-\frac{|F|^2 - \overline{X}^2}{2\sigma_Y^2}} d(|F|^2) - (\overline{X}^2 + \sigma_Y^2)^2 .$$

Making tne same change of variables used in obtaining the mean and again using Eq. (53) we find that the variance of tne power is

$$\begin{aligned} \mathrm{Var}(|F|^2) &= 2\sigma_Y^4 \\ &\approx \frac{2\Delta^4}{9} \left(\sum_n a_n^2\right)^2 . \end{aligned} \tag{55}$$

Comparing the approximation (55) with the exact (to within the small phase error approximation) expression for the variance of tne power given by Eq. (13) and noting that for the special case we are considering $H_o(u) = \sum a_n^2 \cos[2d_n(u - u_s)] = \sum a_n^2$, we see that the exact expression contains several terms in addition to those in the approximation (55). Similarly to our discussion on p. 35 of the apparent discrepancy at a pattern null between the exact expression for the variance, and the expression obtained from the Central Limit Theorem and the probability distribution of the amplitude, it can be shown (see Appendix A) that the extra terms can be neglected for large arrays and so the two results agree.

For the second case, we consider the approximation (51), and begin by noting that the error-free pattern has a null at the directions we are considering. Since $\sum a_n^2 \cos[2d_n(u - u_s)] = -\sum a_{n'}^2 \cos[2d_n(u - u_s)] = -1$ and hence

$$\cos[2d_n(u-u_s)] = \cos\left\{2\left[\frac{N-1}{2} - (n-1)\right] m\pi\right\}$$

$$= \cos[(N-1)\,m\pi]$$

$$= -1$$

so that

$$(N-1)\,m\pi = \pm\,\pi,\ \pm\,3\pi,\ \pm\,5\pi,\ \cdots$$

and

$$(N-1)m = \pm 1,\ \pm 3,\ \pm 5,\ \cdots\ .$$

But then

$$\cos[d_n(u-u_s)] = \cos\left\{\left[\frac{N-1}{2} - (n-1)\right] m\pi\right\}$$

$$= (-1)^{(n-1)m}\cos\left(\frac{N-1}{2}\,m\pi\right)$$

$$= 0$$

since $(N-1)\,m$ is odd and so

$$F_o(u) = \sum_n a_n \cos[d_n(u-u_s)] = 0\ .$$

Since the error-free pattern has a null, the mean of the real part of the field, $\overline{X}$, equals zero. Thus $|F| = |X|$ and the probability density of the amplitude and power are

$$p(|F|) = \frac{1}{\sigma_X}\left(\frac{2}{\pi}\right)^{1/2} e^{-\frac{|F|^2}{2\sigma_X^2}}$$

and

$$p_{|F|^2}(|F|^2) = \frac{1}{(2\pi)^{1/2}\sigma_X} \frac{e^{-\frac{|F|^2}{2\sigma_X^2}}}{|F|}$$

respectively. These density functions are identical with those obtained for the first case by the substitutions $\overline{X} \longleftrightarrow 0$, $\sigma_Y^2 \longleftrightarrow \sigma_X^2$, so that the mean and variance of the power are

$$\overline{|F|^2} = \sigma_X^2$$

$$\approx \frac{\Delta^2}{3} \sum_n a_n^2$$

and

$$\mathrm{Var}(|F|^2) = 2\sigma_X^4$$

$$\approx \frac{2\Delta^4}{9}\left(\sum_n a_n^2\right)^2 \qquad (56)$$

respectively. Comparing the approximation (56) with the more exact expression given by Eq. (13) for the variance, noting that all the terms in the approximation (13) containing $F_o(u)$ are zero because of the null in the error free pattern and that $|H_o(u)|^2 = \left(\sum a_n^2\right)^2$, and recalling that $\sum a_n^4$ is of the order of $1/N$ compared to $\left(\sum a_n^2\right)^2$ for large N, we see that the two expressions for the variance are in agreement.

3. NUMERICAL RESULTS AND DISCUSSION

Example calculations in this section give results that are compared with results obtained in the analysis of the preceding section. We consider here the distribution of the power at (1) a single null, (2) multiple nulls, and (3) a non-null sidelobe location.

3.1 Single Null

We begin with the distribution of the power at a direction for which the error-free pattern has a perfect null. For concreteness we consider a broadside array of

79 elements with half wavelength spacing and a 40 dB Chebyshev taper. The phases of the array elements are assumed to be set with 8 bit digital phase shifters. The error-free pattern of the array has a null at 20.3989...° between the 13th and 14th sidelobes. Calculating the mean power at this location using Eq. (4) we find

$$\overline{|F|^2} = 0.8072 \times 10^{-6} = -60.9 \text{ dB}$$

and calculating the variance of the power from Eq. (12),

$$\mathrm{Var}(|F|^2) = 0.6351 \times 10^{-12};$$

hence, the standard deviation of the power equals 0.7969×10^{-6}. Note that the standard deviation of the power is approximately equal to the mean power, in agreement with the discussion following Eq. (16). The contribution to the variance of first three terms on the right hand side of Eq. (12) is effectively zero because of the null of the error-free pattern; the contribution of the fourth term (in $|H_o(u)|^2$) equals 0.84×10^{-17}, the fifth term equals 0.65×10^{-12}, and the last term equals 0.17×10^{-13}. Thus, as was shown on p. 20, $|H_o(u)|^2$ is negligible compared to $\left(\sum |w_n|^2\right)^2$ (except at directions halfway between grating lobes), and, as discussed on p. 20 and proved in Appendix A, the term in $\sum |w_n|^4$ is of the order of $1/N$ compared with the term in $\left(\sum |w_n|^2\right)^2$.

We compute the probability distribution of the power at the same location from Eqs. (28) and (29) with the result, $\sigma_X^2 = 0.405 \times 10^{-6}$ and $\sigma_Y^2 = 0.402 \times 10^{-6}$ with the term in $\sum a_n^2$ equal to 0.403×10^{-6} and the term in $\sum a_n^2 \cos(2\psi_n)$ equal to 0.145×10^{-8} and hence negligible compared to the term $\sum a_n^2$ as was shown in the discussion of Eq. (38). The parameter B of the normalized probability distribution, Eq. (34a), is effectively zero because of the pattern null, and the parameter K given by Eq. (34c) is equal to 0.996. Since $K \approx 1$ and $\overline{X} = 0$, we expect the probability distribution of the power to be well approximated by the exponential distribution given in Eq. (44) with a mean of $\frac{\Delta^2}{3} \sum a_n^2 = 0.8072 \times 10^{-6}$ and a variance equal to the square of the mean, 0.6516×10^{-12}. Note that the variance of the power calculated assuming the exponential distribution of the power is quite close to the value of 0.6351×10^{-12} obtained from the exact expression for the variance.

To verify the above theoretical results we wrote a computer program to randomly perturb the phases of an equispaced array and calculate the resulting power at a specified angle, given the number of elements, the spacing, the amplitude taper, and the number of bits of the phase shifters. Each phase is independently perturbed from its error-free value by a random number taken from a uniform distribution in the interval $[-\pi/2^{N_{bit}}, \pi/2^{N_{bit}}]$. A sample size of 1000 patterns was

taken for a 79 element, half wavelength spacing, 40 dB Chebyshev tapered array. The mean power at the location $\theta = 20.3989\ldots°$ was found to be 0.8043×10^{-6} and the variance to be 0.6167×10^{-12}, in close agreement with the theoretical results. In Figure 1 we show a computer generated plot (obtained using the IMSL routine USPC) of the sample cumulative probability distribution and the theoretical distribution of Eq. (47). (The letter M indicates that both curves share the same print location.) The close fit of the two curves is apparent. To further check the agreement of the sample and theoretical cumulative distributions, a Kolmogorov-Smirnov test (described in Reference 14) was performed using the IMSL routine NKS1. The maximum absolute difference between the sample cumulative distribution and the theoretical exponential distribution, D_n (n = size of sample, here equal to 1000) was found to be 0.0228, the statistic $z = \sqrt{n}\, D_n = 0.72$, and the probability of obtaining a value of z equal to or greater than 0.72 was found to be 0.68 so that there is no reason to reject the hypothesis that the exponential distribution is the underlying population distribution of the power.

A similar set of calculations was performed for a null that was imposed on the 79 element, 40 dB Chebyshev pattern by small phase perturbations, using an iterative technique described in Reference 15. The results do not differ in any significant way from those obtained for the "natural" null and need not be described here.

For reference purposes it is useful to have a plot of the normalized exponential cumulative probability distribution of the power at a direction for which the error-free pattern has a null. In Figure 2 we have plotted the cumulative probability distributions [Eq. (48)]

$$C_M(\rho^2) = \left[1 - \exp(-\rho^2)\right]^M$$

for M = 1 to 3 as a function of $10 \log_{10}(\rho^2)$ where ρ^2 is the power normalized by $2\sigma^2$ with

$$2\sigma^2 = \left[1 - \left(\frac{\sin\Delta}{\Delta}\right)^2\right] \sum_n a_n^2 \Big/ \left(\sum_n a_n\right)^2$$

$$\approx \frac{\Delta^2}{3} \sum_n a_n^2 \Big/ \left(\sum_n a_n\right)^2, \quad \Delta = \frac{\pi}{2^{N_{bit}}}$$

The curve M = 1 is the exponential cumulative probability distribution. (The curves for other values of M are relevant to the discussion below of multiple nulls.) As an example of the use of this plot, in the case considered above $N_{bit} = 8$ so that

(Because of the length of Reference 14, References 14 and 15 will not be listed here. See References, page 57.)

$\Delta = \pi/_{256} \approx 0.012$. The sum $\sum a_n^2$ normalized by $\left(\sum a_n\right)^2$ is calculated to be 0.01608 so that $2\sigma^2 = 0.8072 \times 10^{-6} = -60.9$ dB. The probability of a null depth 20 dB below $2\sigma^2$ (-80.9 dB from the mainlobe peak) is then seen to be 0.01, and the probability of a null depth 10 dB below $2\sigma^2$ (-70.9 dB from the mainlobe) is 0.095. The probability of a value of power more than -4 dB below $2\sigma^2$ (that is, more than -64.9 from the mainlobe) is 0.5.

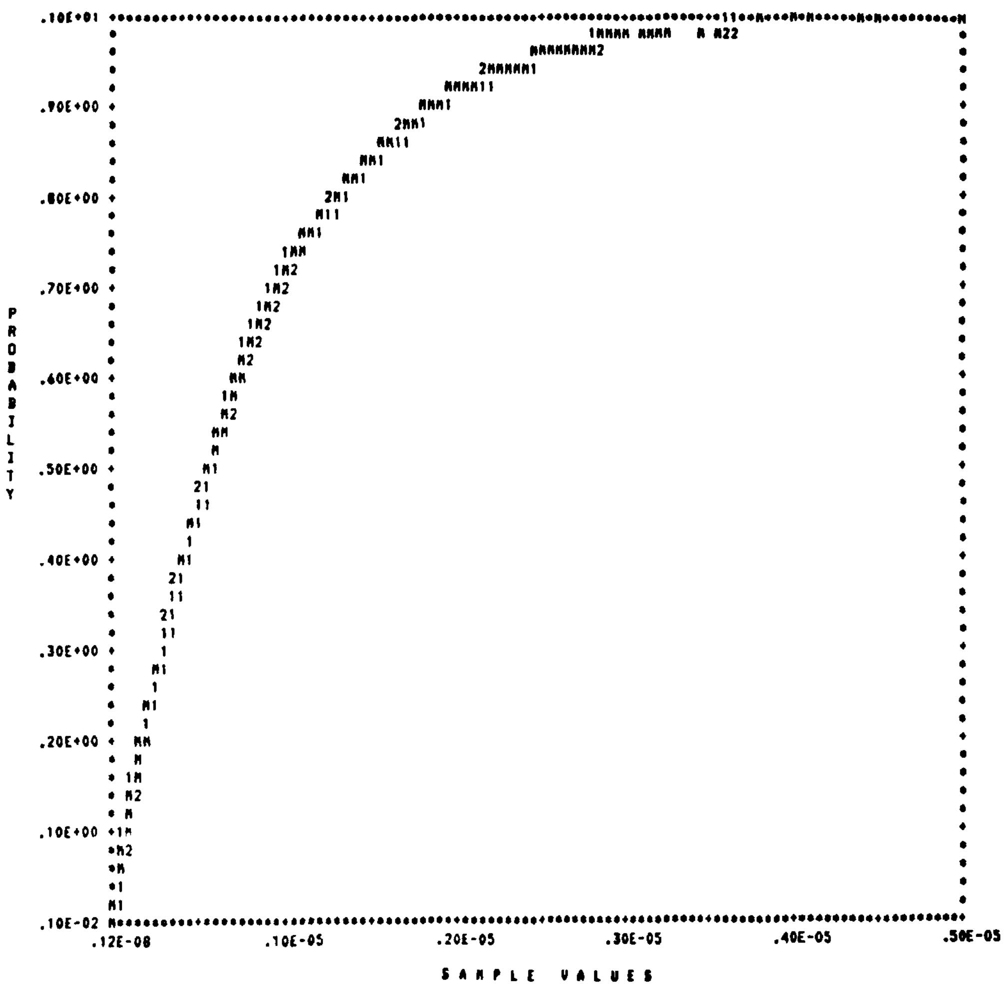

Figure 1. Sample (1) and Theoretical (2) Cumulative Distribution Functions of Power at the Error-free Pattern Null Location $\theta = 20.3989\ldots°$. Half wavelength spacing is used in a 79-element, 40 dB Chebyshev tapered array with phase errors uniformly distributed in $[-\Delta, \Delta]$, $\Delta = \frac{\pi}{256} = 0.012$ (M indicates that both curves share the same print location)

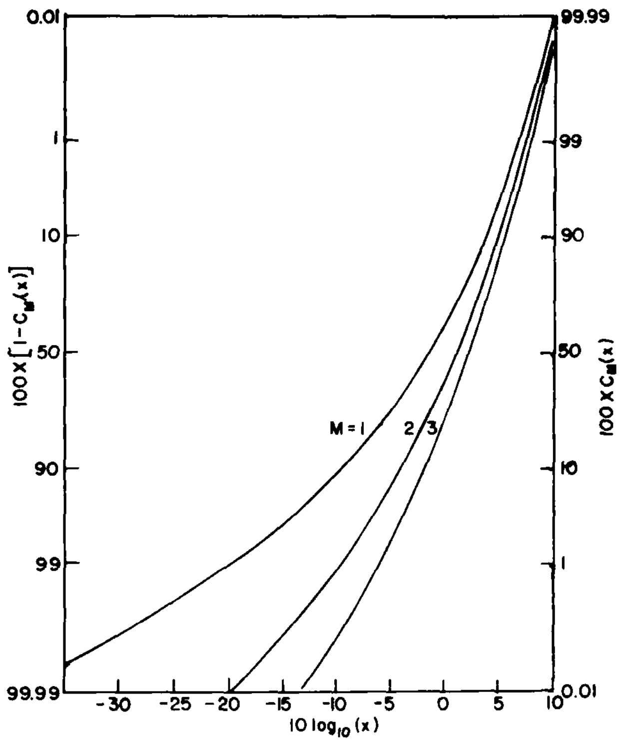

Figure 2. The Cumulative Probability Distribution Function $C_M(\rho^2) = [1 - \exp(-\rho^2)]^M$ for M = 1, 2, and 3

3.2 Multiple Nulls

We next describe computations to compare with the analytic results obtained for the distribution of the least deep null among several directions for each of which the error-free pattern has a perfect null. We showed in the previous section that, if we make the simplifying assumption that the null depths at any two different locations are statistically independent of each other, the cumulative distribution of the least deep null among M locations is given by Eq. (48). In the first computation performed here, we took a sample of 1000 random perturbations of the phases of the 79 element, half wavelength, 40 dB Chebyshev array with $N_{bit} = 8$, and examined the distribution of the least deep null between the locations $\theta = 20.3989...°$ and $21.9620...°$. These locations are the nulls between the 13th and 14th sidelobes, and between the 14th and 15th sidelobes, respectively. The values predicted for the mean and variance of the sample on the basis of Eq. (48) (see Table 2) are

$$\overline{|F|^2} = \frac{3}{2}(2\sigma^2) = \frac{3}{2}(0.8072 \times 10^{-6}) = 0.1211 \times 10^{-5}$$

and

$$\mathrm{Var}(|F|^2) = \frac{5}{4}(2\sigma^2) = 0.8145 \times 10^{-12} .$$

The values calculated from the sample of 1000 were

$$\overline{|F|^2}_{1000} = 0.1123 \times 10^{-5}, \ \mathrm{Var}(|F|^2)_{1000} = 0.7297 \times 10^{-12} ,$$

reasonably close to the predicted values although differing somewhat more in relative terms than the corresponding results for a single null at $\theta = 20.3989\ldots°$ described above. The computer generated plot of the sample and theoretical cumulative distribution functions is shown in Figure 3. Comparing Figure 3 with Figure 1 shows that the fit of the theoretical with the sample cumulative distribution function (CDF) is somewhat poorer for the two-null case especially in the middle portion of the plot. Note that the sample CDF is fairly consistently higher than the theoretical CDF indicating that a greater proportion of the sample values of the least deep null are to be found in a lower part of the range of power than is predicted by the theoretical distribution. A Kolmogorov-Smirnov test was performed on the sample. The maximum absolute difference between the sample and theoretical CDF's, D_n, was 0.07, the statistic $z = 2.21$, and the probability of obtaining a value of z equal to or greater than 2.21 was found to be 0.0001 indicating that the hypothesis that the underlying population distribution of the sample is described by the theoretical distribution given by Eq. (48), should be rejected.

A similar computation was performed for the case of two nulls imposed on the 79 element, 40 dB Chebyshev pattern by small phase perturbations at the locations $\theta = 20.78°$ and $21.2°$ (approximately the left -3 dB point and the sidelobe peak of the 13th sidelobe). The sample mean and variance were

$$\overline{|F|^2}_{1000} = 0.9044 \times 10^{-6}, \ \mathrm{Var}(|F|^2)_{1000} = 0.7196 \times 10^{-12} .$$

A plot of the sample and theoretical CDF's is shown in Figure 4. Here the divergence between the two distributions is clearly apparent, the sample distribution being significantly higher than the theoretical distribution over almost the entire range. This of course is reflected in the considerably lower value of the sample mean compared with the theoretical mean. The Kolmogorov-Smirnov test gave $D_n = 0.2080$, $z = 6.58$, and Prob $z \geq 6.58 \approx 0.0000$.

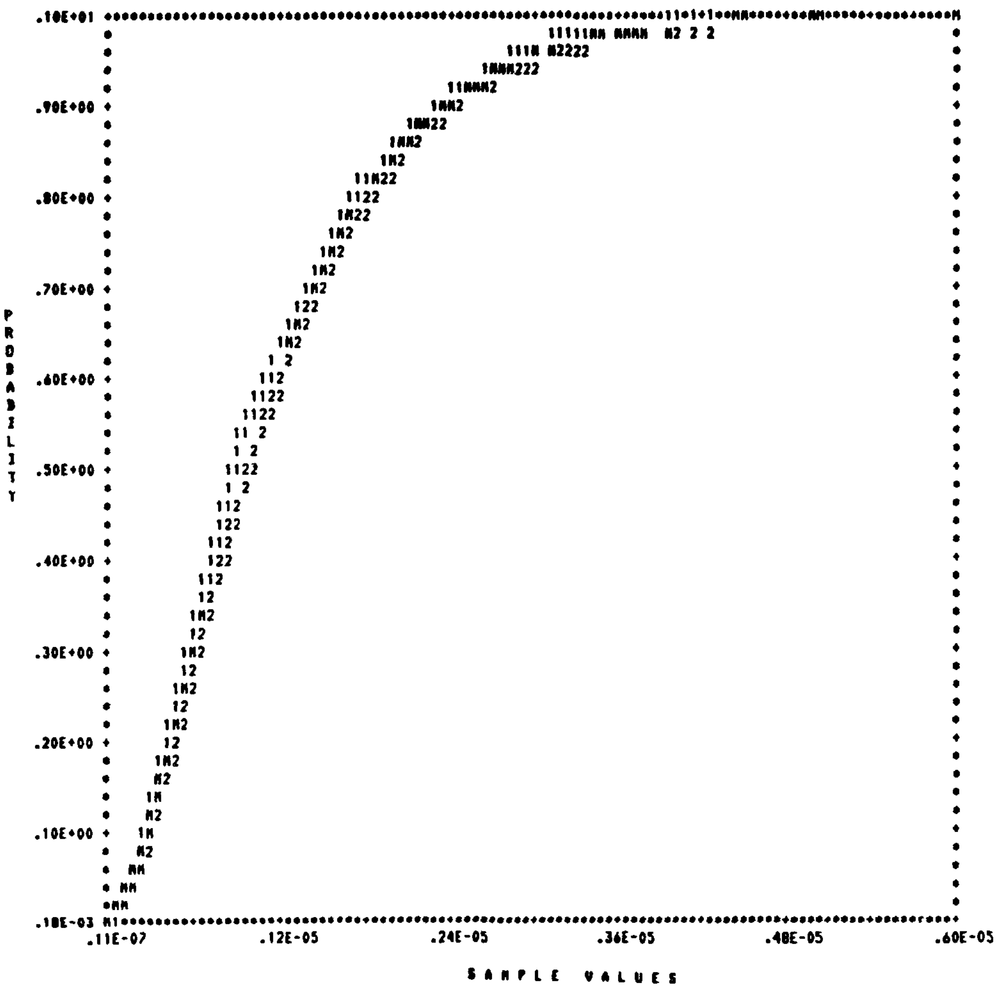

Figure 3. Sample (1) and Theoretical (2) Cumulative Distribution Functions of the Maximum Value of Power at the Two Error-free Pattern Null Locations θ = 20.3989... ° and 21.9620... °. Half wavelength spacing is used in a 79-element 40 dB Chebyshev tapered array with phase errors uniformly distributed in $[-\Delta, \Delta]$, $\Delta = \frac{\pi}{256} = 0.012$ (M indicates that both curves share the same print location)

The fact that the sample CDF is consistently higher than the theoretical CDF indicates, as mentioned above, that a significantly greater proportion of the sample values for the least deep null are below those predicted by the theoretical distribution, and suggests that the null depths at closely spaced locations of the pattern are not independent of each other as was assumed in deriving the theoretical CDF given by Eq. (48). If the null depth is low at one location it tends to be low at the other

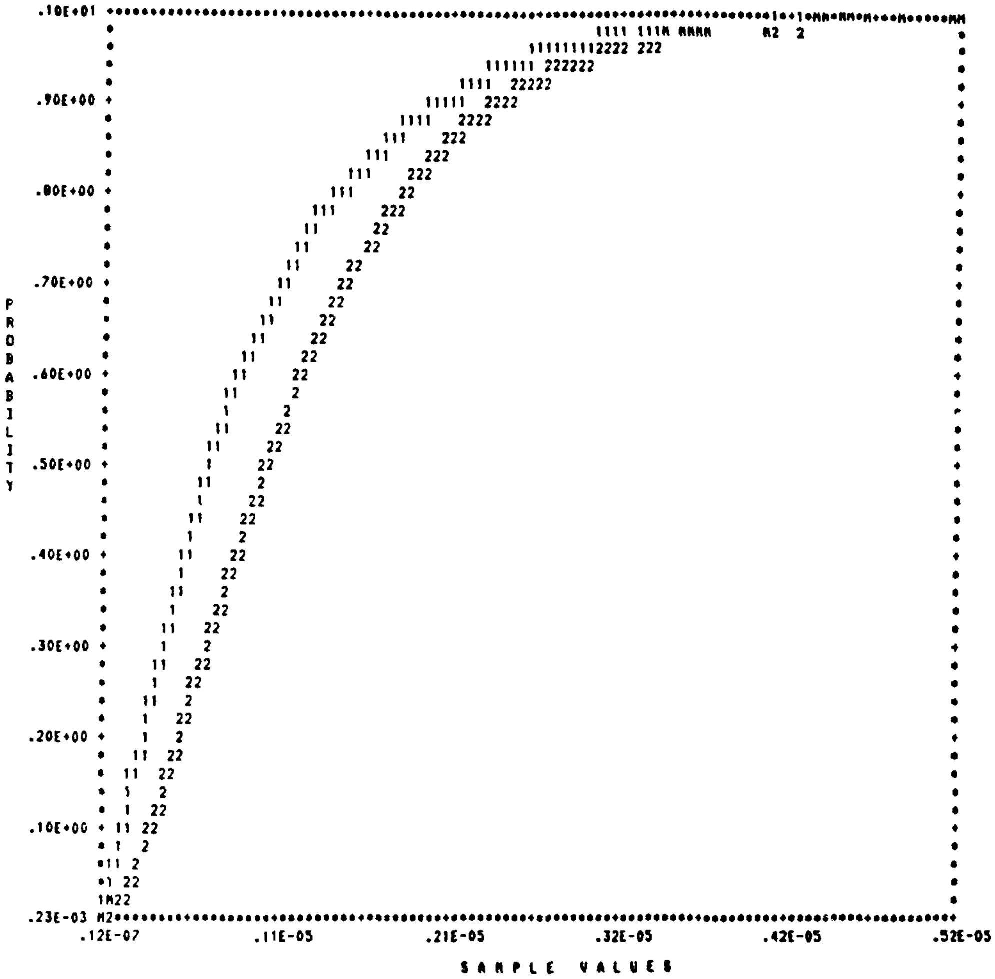

Figure 4. Sample (1) and Theoretical (2) Cumulative Distribution Functions of the Maximum Value of Power at the Two Error-free Pattern Imposed Null Locations θ = 20.78° and 21.2°. Half wavelength spacing is used in a 79-element, 40 dB Chebyshev tapered array with phase errors uniformly distributed in $[-\Delta, \Delta]$, $\Delta = \frac{\pi}{256} = 0.012$ (M indicates that both curves share the same print location)

location as well. As a check on the validity of this hypothesis, another calculation was performed in which nulls were imposed on the basic 79 element, 40 dB Chebyshev pattern at the widely spaced locations θ = 12° and 72°, and a sample taken of 1000 values of the least deep null among these two locations. The sample mean and variance were

$$\overline{|F|^2}_{1000} = 0.1177 \times 10^{-5} \text{ (-59.3 dB)}, \ \mathrm{Var}(|F|^2) = 0.7747 \times 10^{-12},$$

and the plot of the sample and theoretical CDF's is shown in Figure 5. The much closer fit of the sample and theoretical CDF's than in Figure 4 is apparent and is reflected in the Kolmogorov-Smirnov test which gave

$$D_n = 0.025, \ z = 0.776, \ \text{Prob } z \geqq 0.776 = 0.58,$$

indicating consistency of the sample and theoretical CDF's.

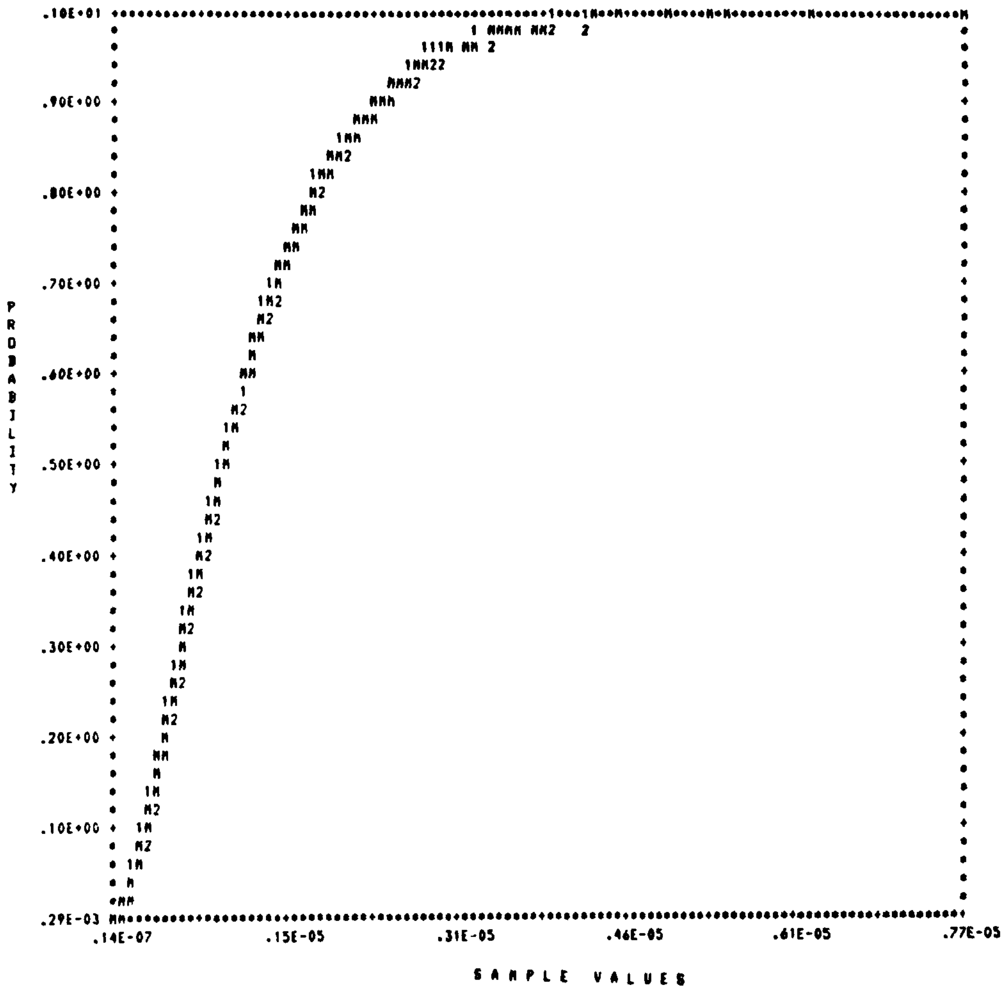

Figure 5. Sample (1) and Theoretical (2) Cumulative Distribution Functions of the Maximum Value of Power at the Two Error-free Pattern Imposed Null Locations $\theta = 12°$ and $72°$. Half wavelength spacing is used in a 79-element, 40 dB Chebyshev tapered array with phase errors uniformly distributed in $[-\Delta, \Delta]$, $\Delta = \frac{\pi}{256} = 0.012$ (M indicates that both curves share the same print location)

The high statistical correlation between the null depths at neighboring locations has interesting implications for nulling with phased arrays employing digital phase shifters when closely spaced nulls are desired. It means that the assumption of independence of null depths gives an overly conservative estimate of the probability that the null depths at two or more closely spaced locations (for each of which the error-free pattern has very deep nulls) will both be less than a given level. Suppose, for example, that the probability of a single null being 20 dB below $2\sigma^2$ is 0.1. Then under the assumption of independence we would conclude that the probability that two adjacent nulls were both less than 20 dB below $2\sigma^2$ was 0.01 (see Figure 2 in which the theoretical CDF, Eq. (48), is plotted for n = 1, 2, and 3). In fact, however, things are not all that bad and the probability that both nulls are less than 20 dB below $2\sigma^2$ may well be closer to 0.1 than 0.01. To illustrate the fact that the one null theoretical CDF may give a better fit to the two null sample data than the two null theoretical CDF, in Figure 6 we plot the sample CDF of Figure 4 (corresponding to nulls at 20.78° and 21.2°) along with the single null theoretical CDF. The single null theoretical CDF is clearly closer to the sample CDF than is the two-null CDF (z = 2.65 under the assumption of the one-null CDF whereas z = 6.58 for the two-null CDF). We also performed a similar computation for the case of three nulls at θ = 20.78°, 21.2°, and 21.8°, and found that the one-null theoretical CDF gave a better fit to the three-null sample distribution than either the three-null or two-null theoretical CDF's (z = 4.59 under the assumption of the one-null theoretical CDF, whereas z = 9.43 for the three-null CDF and 4.89 for the two-null CDF). It would be interesting to study in detail the quantitative dependence of the correlation of null depths on the angular separation of the null locations and on other factors such as the magnitude of the quantization error, the number and spacing of the elements, and the amplitude taper. Such an investigation, however, lies beyond the scope of the present study and we limit ourselves here to drawing attention to the fact that such a correlation is important in determining the null depths in the error pattern at closely spaced null locations.

3.3 Non-null Sidelobe Location

To verify the results obtained in the previous section for the theoretical distribution of power at a non-null sidelobe location not halfway between grating lobes [see the discussion of Eq. (38)], a sidelobe location of θ = 20.1° was chosen in the 13th sidelobe of the 79 element, 40 dB Chebyshev pattern. In Figure 7 we show the computer generated plot of the sample CDF based on 1000 samples of phase errors with $N_{bit} = 8$, and the theoretical Rician CDF calculated using the asymptotic form (43). For this value of θ, the parameter $\alpha = 2\sigma^2\overline{X} = 8.99$ so that the asymptotic form is an excellent approximation to the exact integral of the Rician probability

distribution function. The close fit of the sample and theoretical distribution functions is apparent and is corroborated by the results of the Kolmogorov-Smirnov test: $D_n = 0.02$, $z = 0.63$, and Prob $z \geq 0.63 = 0.82$.

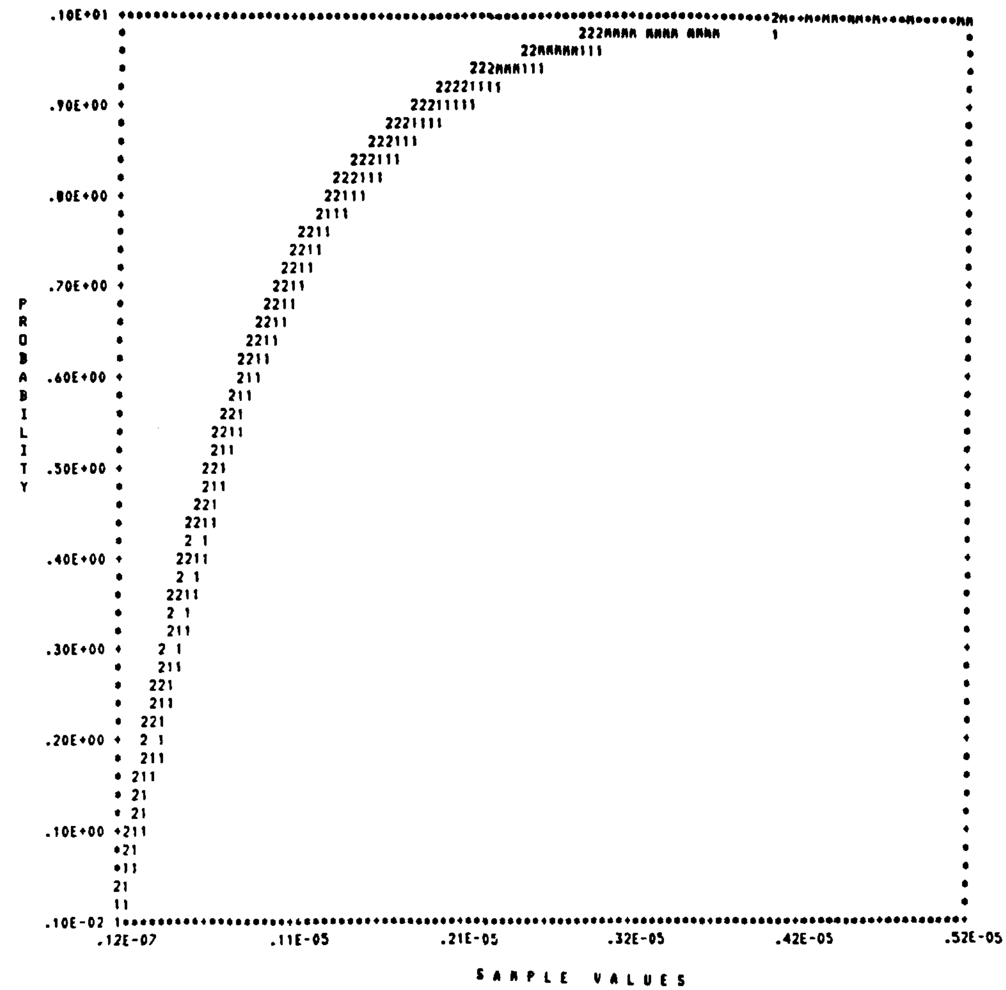

Figure 6. Sample (1) and Theoretical (2) Cumulative Distribution Functions of the Maximum Value of Power at the Two Error-free Pattern Imposed Null Locations $\theta = 20.78°$ and $21.2°$; Theoretical CDF is for One Error-free Pattern Null Location. Half wavelength spacing is used in a 79-element, 40 dB Chebyshev tapered array with phase errors uniformly distributed in $[-\Delta, \Delta]$, $\Delta = \frac{\pi}{256} = 0.012$ (M indicates that both curves share the same print locations)

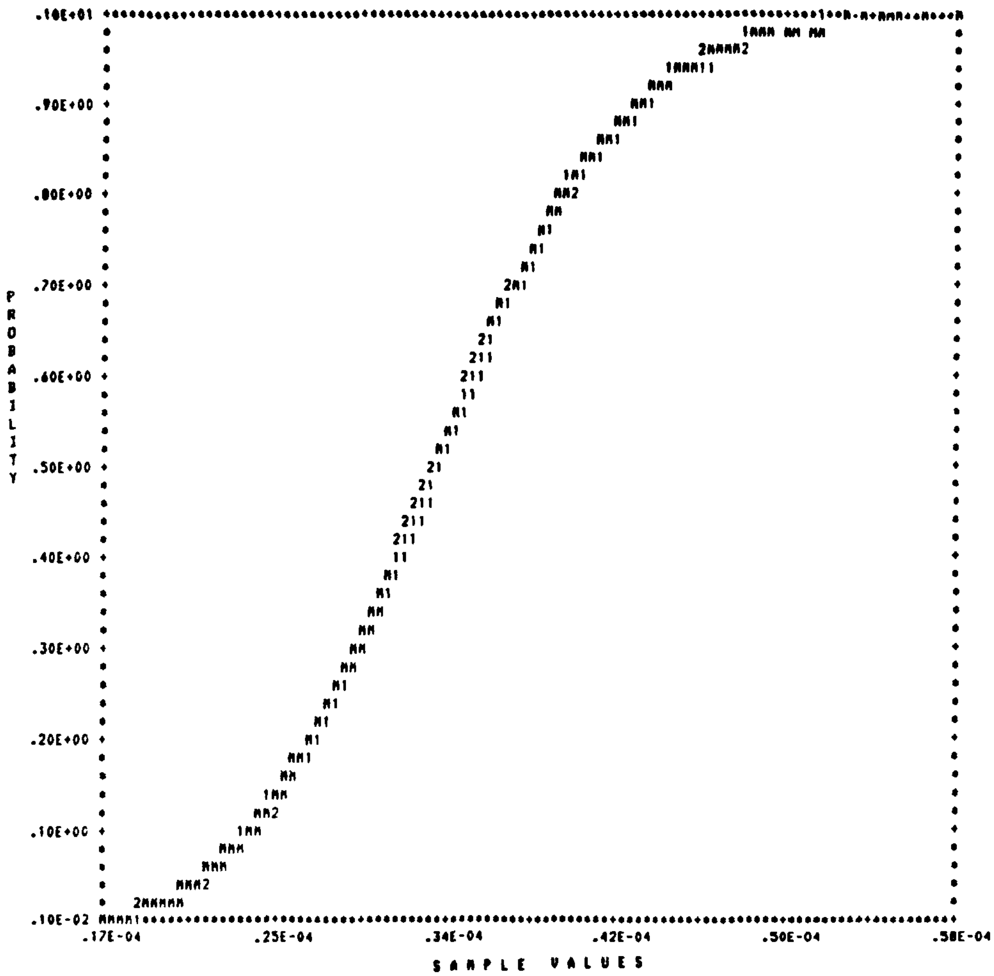

Figure 7. Sample (1) and Theoretical (2) Cumulative Distribution Functions of Power at the Error-free Pattern Non-null Location $\theta = 20.1°$. Half wavelength spacing is used in a 79-element, 40 dB Chebyshev tapered array with phase errors uniformly distributed in $[-\Delta, \Delta]$, $\Delta = \frac{\pi}{256} = 0.012$ (M indicates that both curves share the same print location)

For the case in which a non-null sidelobe location is halfway between grating lobes, the theoretical mean, variance, and CDF, are given respectively by the approximations (54) and (55), and by Eq. (52). To compare with these theoretical results, a sample distribution was generated starting with a 79 element, one wavelength spacing, 40 dB Chebyshev taper broadside array, and introducing random phases uniformly distributed in the interval $[-\pi/2^{N_{bit}}, \pi/2^{N_{bit}}]$ with $N_{bit} = 8$. The

sample mean and variance of the power at the locations $\theta = 30°$, halfway between grating lobes, based on a sample size of 1000, were 0.1019×10^{-3} and 0.1345×10^{-11} respectively as compared with the theoretical values of 0.1018×10^{-3} (-39.96 dB) and 0.1303×10^{-11}. In Figure 8 we show computer generated plots of the sample and theoretical CDF's. The two plots appear to agree well but this apparent agreement is not corroborated by the Kolmogorov-Smirnov test which gives $D_n = 0.075$, $z = 2.37$, and Prob $z = \geq 2.37 = 0.0000$. The explanation of this discrepancy between the result of the Kolmogorov-Smirnov test and the close agreement of the theoretical and sample mean, variance, and distribution function plots, lies in the fact that the theoretical results are based on the approximations (50) which preclude any variation of the real part of the field. As a result, the theoretical power distribution has no values of power less than

$$X^2 \approx \left(1 - \frac{\Delta^2}{3}\right) F_o^{\,2}(u) .$$

In fact, however, the real part of the field has a very small but non-zero variance (if the variance of the real and imaginary parts of the field are calculated using Eqs. (28) and (29) we find that $\sigma_X^2 = 0.8104 \times 10^{-11}$ and $\sigma_Y^2 = 0.8072 \times 10^{-6}$) so that a number of the sample values of power are slightly smaller than the theoretical lower limit. This results in a significant difference between the sample and theoretical distribution functions at the very beginning of the range of power which, although not apparent in the plots, makes itself felt as a sufficiently large value of D_n to result in a Kolmogorov-Smirnov rejection of the hypothesis that the sample of distribution of power is well described by the theoretical distribution. Apart from this difference in behavior at the lower limit of the range in power, the theoretical and sample distributions agree well as is evidenced by the plots and by the respective values for the mean and variance.

4. CONCLUSIONS

In this report we have analyzed the statistical distribution of sidelobe power of a linear array of isotropic, equispaced elements whose excitation coefficients are subject to random phase errors. Since our study was motivated by the desire to describe the errors introduced in the array pattern by the use of digital phase shifters to set the excitation phases, the random phase errors were assumed throughout to be uniformly distributed in the interval $[-\pi/2^{N_{bit}}, \pi/2^{N_{bit}}]$ where N_{bit} is the number of bits in the phase shifters.

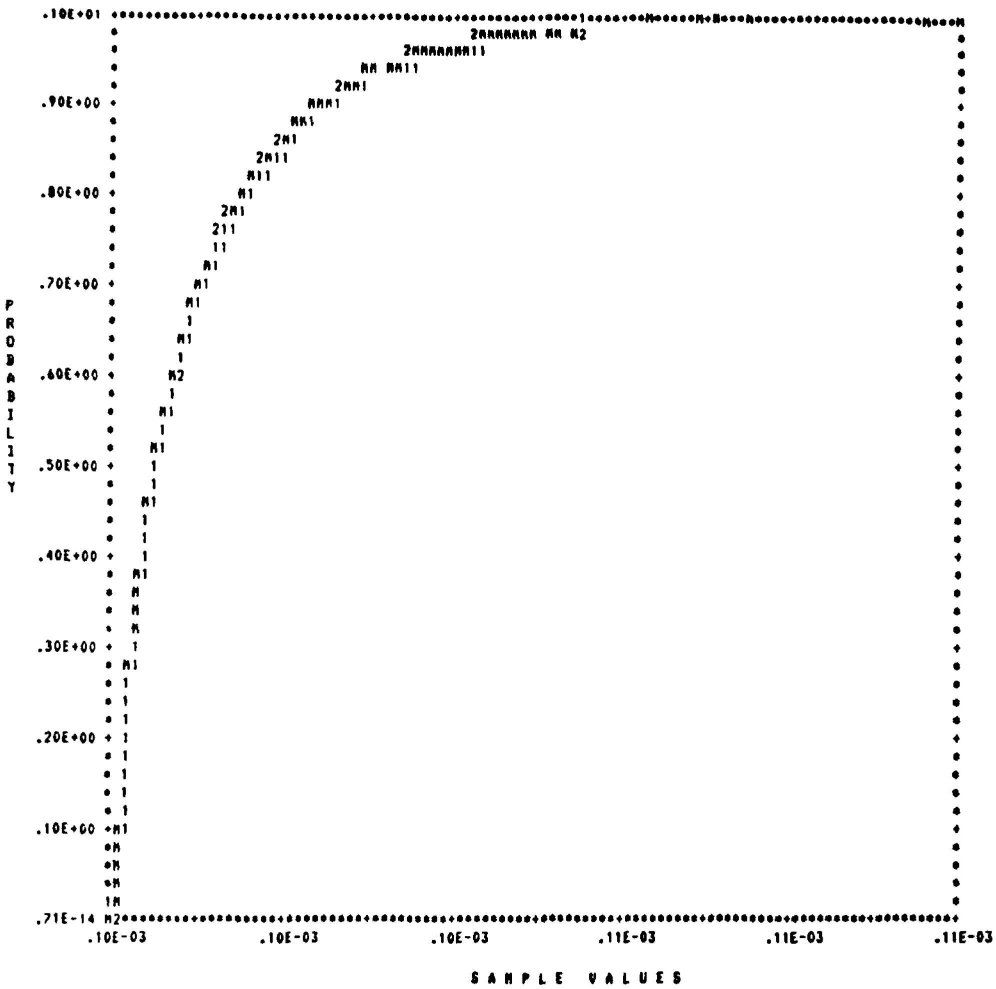

Figure 8. Sample (1) and Theoretical (2) Cumulative Distribution Functions of the Power at the Location $\theta = 30°$, Midway Between Grating Lobes of the Error-free Pattern. One wavelength spacing is used in a 79-element, 40 dB Chebyshev tapered array with phase errors uniformly distributed in $[-\Delta, \Delta]$, $\Delta = \frac{\pi}{256} = 0.012$ (M indicates that both curves share the same print location)

We began by deriving general expressions for the mean and variance of the power at an arbitrary pattern location, and obtained small phase error approximations valid for $N_{bit} > 4$. We showed that for large arrays, the variance of the power at a sidelobe location for which the error-free pattern has a null is approximately equal to the square of the mean of the power.

We then derived a general expression for the probability distribution of the amplitude and power of the error field under the assumptions that the error-free field was real and that the number of elements in the array was sufficiently large for the Central Limit Theorem to be applied. This general expression was then specialized to the case of small phase errors and ordinary sidelobe locations (locations not halfway between grating lobes). For this case the distribution of sidelobe amplitude is described by the Rice-Nakagami distribution. At a pattern location for which the error-free pattern has a null, the Rice-Nakagami distribution reduces to the Rayleigh distribution, and the probability distribution of the power is then described by the exponential distribution. The expressions for the mean and variance of the power obtained using the exponential distribution were compared with the general expression for the mean and variance of the power obtained earlier, and the two sets of expressions shown to be consistent under the assumption of large arrays used in deriving the probability distribution of the power. Assuming statistical independence of the depth of null at two or more sidelobe locations for which the error-free pattern has a null, an expression for the statistical distributions of the least deep null among several locations was obtained by multiplying together the distributions for the individual nulls. Finally, the distribution of power at locations halfway between grating lobes was obtained starting with expressions for the variance of the real and imaginary parts of the field, and showing that at such points the variance of either the real or imaginary part of the field is approximately equal to zero.

Following the theoretical analysis, Monte Carlo-type computer simulations were performed to compare with the theoretical results. Extensive use was made of the Kolmogorov-Smirnov test to test the hypothesis that the sample distribution of power generated by the simulation could be regarded with high probability as being drawn from the corresponding underlying distribution obtained theoretically. In general, close agreement between the simulation and theoretical results was obtained. However, the assumption of statistical independence of null depths at several locations for which the error-free pattern has a null was shown to be invalid for closely spaced nulls. The correlation between the null depths at closely spaced locations increases the probability of the shallowest null having a given minimum depth from the independent-multiple null probability to a value close to the probability of a single null of that depth.

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4. Cauchy's Inequality:

$$\left[\sum_{k=1}^{n} a_k b_k\right]^2 \le \sum_{k=1}^{n} a_k^2 \sum_{k=2}^{n} b_k^2 \qquad \text{(equality for } a_k = c\,b_k \text{, } c \text{ constant).}$$

See Abramowitz, M. (1972) Elementary analytical methods, Chap. 3 of Handbook of Mathematical Functions, M. Abramowitz and I. Stegun, (Eds.), Dover, N.Y.

5. Chebyshev's Inequality:

$$\text{If } a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n$$

$$b_1 \ge b_2 \ge b_3 \ge \cdots \ge b_n$$

$$n \sum_{k=1}^{n} a_k b_k \ge \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

See Abramowitz, M.[4].

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14. The Kolmogorov-Smirnov test provides a method for testing the hypothesis that a given sample of size n is derived from an underlying population with a specified cumulative distribution function (CDF). The test is performed by calculating the CDF based on the actual sample, comparing the values of the sample CDF with the hypothetical CDF, and determining the largest absolute difference, D_n, between the sample and hypothetical CDF's. Given D_n, the statistic $z = nD_n$ is computed and the probability calculated that z is equal to or larger than the value actually obtained. The larger the maximum difference between the sample and hypothetical CDF's, the larger will be the value of z and the smaller the probability that random sampling alone can account for the observed difference between the sample hypothetical CDF's. For further details see Kendall, M.G., and Stuart, A. (1967) The Advanced Theory of Statistics, Vol. 2, Hafner, New York, pp. 450-461; Gnedenko, B.S. (1967) The Theory of Probability, Chelsea, New York, pp. 450-451; and Lindgren, B.W. (1967) Statistical Theory, MacMillan, New York, pp. 300-304.

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Appendix A

Proof of Theorems Used in the Analysis

In this appendix we prove two results used in the analysis section of the report. The first result, made use of on pp. 20 and 35, is that for any taper of the amplitude $|w_n|$ of the element excitation coefficients describable by a polynomial

$$\frac{\sum_{n=1}^{N} |w_n|^4}{\left(\sum_{n=1}^{N} |w_n|^2\right)^2} \propto \frac{1}{N} . \tag{A1}$$

To prove (A1) we start with the assumption that the amplitudes of the excitation coefficients can be expressed as a polynomial of degree M,

$$|w_n| = \sum_{k=0}^{M} a_k \left(1 - 2\frac{n-1}{N-1}\right)^k , \quad n = 1, 2, \ldots, N \tag{A2}$$

with M and the a_k independent of N. Eq. (A2) in effect defines a family of element amplitudes, each member of the family corresponding to a particular value of N. The form of the argument of the polynomial is chosen to equal +1 and -1 for n = 1 and N respectively. Expanding Eq. (A2) and collecting terms in $\left(\frac{n-1}{N-1}\right)^k$ we can write

$$|w_n| = \sum_{k=0}^{M} b_k \left(\frac{n-1}{N-1}\right)^k . \tag{A3}$$

Then $|w_n|$ raised to some integral power P is in turn a polynomial, of degree MP

$$|w_n|^P = \sum_{k=0}^{MP} c_k \left(\frac{n-1}{N-1}\right)^k$$

and

$$\sum_{n=1}^{N} |w_n|^P = \sum_{n=1}^{N} \sum_{k=0}^{MP} c_k \left(\frac{n-1}{N-1}\right)^k$$

$$= \sum_{k=0}^{MP} c_k \sum_{n=1}^{N} \left(\frac{n-1}{N-1}\right)^k$$

$$= \sum_{k=0}^{MP} c_k \frac{1}{(N-1)^k} \sum_{n=1}^{N-1} n^k .$$

Now $\sum_{n=1}^{N-1} n^k$ is a polynomial of degree $k+1$ in $N-1$,

$$\sum_{n=1}^{N-1} n^k = d_o + d_1 (N-1) + \cdots + d_{k+1} (N-1)^{k+1} .$$

Hence for large N

$$\frac{1}{(N-1)^k} \sum_{n=1}^{N-1} n^k \approx d_{k+1} (N-1) \approx d_{k+1} N$$

and

$$\sum_{n=1}^{N} |w_n|^P \approx \sum_{k=0}^{MP} c_k d_{k+1} N = N \left(\sum_{k=0}^{MP} c_k d_{k+1} \right);$$

that is, $\sum_{n=1}^{N} |w_n|^P$ is proportional to N regardless of the value of P. (The proportionality constant does, of course, depend on P.) It follows that $\sum_{n=1}^{N} |w_n|^4$ is proportional to N and $\left(\sum_{n=1}^{N} |w_n|^2 \right)^2$ is proportional to N^2 so that

$$\frac{\sum_{n=1}^{N} |w_n|^4}{\left(\sum_{n=1}^{N} |w_n|^2\right)^2} \propto \frac{N}{N^2} = \frac{1}{N}$$

as claimed.

The second result, used in the discussion of the approximation (55), is that at a pattern direction midway between grating lobes,

$$u = u_s \pm m\pi, \text{ m odd} \tag{A4}$$

for which

$$H_o(u) = \sum_{n=1}^{N} a_n^2 \cos[2 d_n (u - u_s)] = + \sum_{n=1}^{N} a_n^2, \tag{A5}$$

the general expression (13) for the variance of the power is equal to

$$\frac{2\Delta^4}{9} \left(\sum_{n=1}^{N} |w_n|^2\right)^2 \tag{A6}$$

plus terms of the order of 1/N and higher compared to (A6). For convenience we give (13) again here:

$$\mathrm{Var}\left(|F(u)|^2\right) \overset{\Delta<<1}{\approx}$$

$$\frac{2\Delta^2}{3}\left(1 - \frac{7}{15}\Delta^2\right)\left(\sum_{n=1}^{N} |w_n|^2\right)|F_o(u)|^2$$

$$- \frac{8}{45}\Delta^4 \,\mathrm{Re}\,[G_o(u)\, F_o^*(u)] - \frac{2\Delta^2}{3}\left(1 - \frac{3}{5}\Delta^2\right)\mathrm{Re}\,[H_o(u) F_o^*(u)]$$

$$+ \frac{\Delta^4}{9}|H_o(u)|^2 + \frac{\Delta^4}{9}\left(\sum_{n=1}^{N} |w_n|^2\right)^2 - \frac{2\Delta^4}{15}\sum_{n=1}^{N} |w_n|^4 \tag{13}$$

where, under the assumption of real error-free fields used in deriving the approximation (55)

$$F_o(u) = \sum_{n=1}^{N} |w_n| \cos[d_n (u - u_s)] \tag{A7}$$

and

$$G_o(u) = \sum_{n=1}^{N} |w_n|^3 \cos [d_n (u - u_s)] \tag{A8}$$

with

$$d_n = \frac{N-1}{2} - (n-1) .$$

Of the six terms on the RHS of (13), the fourth and fifth are equal for the special case [Eq. (A5)] we are considering, while the last is of the order of 1/N compared to the sum of the fourth and fifth terms because of (A1). Hence we focus our attention on the first three terms of the approximation (13). The first and third terms combine to give

$$\frac{4}{45} \Delta^4 \left(\sum_{n=1}^{N} |w_n|^2 \right) F_o^2(u) .$$

Now from Eq. (A4)

$$2 d_n (u - u_s) = \pm\ 2\left[\frac{N-1}{2} - (n-1)\right] m\pi$$

and hence

$$\cos [2 d_n (u - u_s)] = \cos [(N-1)\, m\pi]$$

so that, for Eq. (A5) to hold, $(N - 1)m$ must be an even integer. (Since m is odd this implies that for the case we are considering there are an odd number of array elements.) But then

$$
\begin{aligned}
F_o(u) &= \sum_{n=1}^{N} |w_n| \cos [d_n (u - u_s)] \\
&= \sum_{n=1}^{N} |w_n| \cos \left\{ \left[\frac{N-1}{2} - (n-1) \right] m\pi \right\} \\
&= \sum_{n=1}^{N} |w_n| (-1)^{(n-1)m} \cos \left(\frac{N-1}{2} m\pi \right) \\
&= \pm \sum_{n=1}^{N} (-1)^{n-1} |w_n|
\end{aligned}
\qquad \text{(A9)}
$$

or

$$
F_o(u) = \pm [|w_1| - |w_2| + |w_3| - + \cdots + |w_N|] .
$$

As above in proving the proportionality (A1) we assume that the amplitudes of the element excitation coefficients can be expressed as the polynomial in Eq. (A2) and hence, using Eq. (A3)

$$
\begin{aligned}
F_o(u) &= \pm \sum_{n=1}^{N} (-1)^{n-1} |w_n| \\
&= \pm \sum_{n=1}^{N} (-1)^{n-1} \sum_{k=o}^{M} b_k \left(\frac{n-1}{N-1} \right)^k \\
&= \pm \sum_{k=0}^{M} b_k \frac{1}{(N-1)^k} \sum_{n=1}^{N} (-1)^{n-1} (n-1)^k \\
&= \pm \sum_{k=0}^{M} b_k \frac{1}{(N-1)^k} \sum_{n=1}^{N-1} (-1)^n n^k .
\end{aligned}
$$

But $\sum_{n=1}^{N} (-1)^n n^k$ is a polynomial of degree k in N - 1,

$$
\sum_{n=1}^{N-1} (-1)^n n^k = d_o + d_1 (N-1) + \cdots + d_k (N-1)^k
$$

so that for large N

$$
F_o(u) \approx \pm \sum_{k=0}^{M} b_k d_k \qquad \text{(A10)}
$$

that is, $F_o(u)$ for large N behaves as a *constant independent of N*. Hence, using the proportionality (A1)

$$\frac{\frac{4}{45}\Delta^4\left(\sum_{n=1}^{N}|w_n|^2\right) F_o^{\,2}(u)}{\frac{2\Delta^4}{9}\left(\sum_{n=1}^{N}|w_n|^2\right)^2} \propto \frac{1}{N}$$

so that the sum of the first and third terms of approximation (13) is of the order $1/N$ compared to the sum of the fourth and fifth terms.

As far as the second term of (13) is concerned, just as Eq. (A9) follows from Eq. (A7), so

$$G_o(u) = \pm \sum_{n=1}^{N} (-1)^{n-1}\,|w_n|^3$$

and in the same way that we established the approximation (A10), a similar relation holds for $G_o(u)$ with the summation on k having the upper limit 3M instead of M. It follows that the second term is of the order $1/N^2$ compared with the sum of the fourth and fifth terms of approximation (13). Hence, as claimed, for the special case we are considering, the expression (13) for the variance of the power is given by expression (A6) plus terms of the order $1/N$ and higher compared to (A6).

END

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